1. The lifetime (in units, 1 unit = 1000 hours) of a 1-unit-rated light bulb has an Exponential distribution with mean $1/\lambda$. In the past, for several batches of light bulbs, a histogram of the mean lifetimes has an average of 0.90 units and a standard deviation of 0.02 units. Suppose the prior distribution for $\lambda$ is Gamma($\alpha, 1/\beta$). (i) Find $\alpha$ and $\beta$. (ii) In an experiment conducted by the manufacturer, 100 such light bulbs were found to burn for a total of 93 units. What is the posterior probability that the average lifetime of the bulb is less than 0.915 units.

2. Suppose the prior guess of $\theta$ in a $U(0, \theta)$ model is summarized by a Pareto (0.01, 1.7) distribution (i.e., $m = 0.01$ and $a = 1.7$). According to a sample $\{0.2, 0.58, 0.1, 1.5, 2.4, 1.77\}$, what is the posterior probability that $\theta > 4$?

3. Consider an Exponential model with mean $1/\lambda$. Suppose $\lambda$ is Gamma (1, $1/2$). (i) What are the predictive mean and variance of a new observation without any data? (ii) How about them after collecting 10 observations with sum equal to 100?

4. In the lecture, we showed that $Var(X_{n+1}) = E\{Var(X_{n+1}|\theta)\} + Var\{E(X_{n+1}|\theta)\}$. Now, (i) Show that

$$Var(X_{n+1}|X_1, \cdots, X_n) = E\{Var(X_{n+1}|\theta)|X_1, \cdots, X_n\} + Var\{E(X_{n+1}|\theta)|X_1, \cdots, X_n\}.$$  

(ii) Using the above formula, solve (i) and (ii) in Problem 3.