Tutorial 3

1. Given an i.i.d sample \( x_1, \ldots, x_n | \tau \sim N \left( h, \frac{1}{\tau} \right) \) where \( h \) is known and \( \tau \sim Gamma \left( \alpha, \frac{1}{\beta} \right) \).

Using the Bayesian Sequential Updating method to show that

\[
(\tau | x) \sim Gamma \left( \alpha_n, \frac{1}{\beta_n} \right)
\]

where

\[
\alpha_n = \alpha + \frac{n}{2} \quad \text{and} \quad \beta_n = \beta + \frac{1}{2} \sum_{i=1}^{n} (x_i - h)^2.
\]

State each step clearly.

2. Suppose that the results of a certain test are known to be approximately \( N(\mu, 1/\tau) \). Suppose further that your prior belief about \((\mu, \tau)\) is \( Gamma - Normal(1, 1/2; 75, 1/3) \). Next, 31 observations are obtained from the population with sample mean 80 and sample variance \( s^2 = 30 \). Find the marginal posterior distribution of \( \tau \). Find 90% posterior HDR of \( \tau \) using the normal approximation method.

3. Notice that the variance \( \sigma^2 = 1/\tau \). Based on the information in Problem 2, find the exact posterior distribution of \( \sigma^2 \).

4. Given an i.i.d sample \( x_1, \ldots, x_n | \theta \sim Binomial(k, \theta) \) where \( k \) is a known positive integer and 0 < \( \theta < 1 \). Find the conjugate family for \( \theta \). Based on a distribution in the conjugate family, find the posterior distribution of \( \theta \), and then find the posterior mean and variance of \( \theta \).