On Some Hypothesis Testing Problems in Functional Data Analysis

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Outline

Introduction

A GPF test for one-way ANOVA problem
The one-way problem and GPF test
Null distribution approximation and asymptotic power
Simulation and Real data analysis

The $L^2$-norm and $F$-type test for two-way ANOVA Problem
$L^2$-norm test under non-Gaussian assumption
$F$-type test under Gaussian assumption
Simulation and Real data analysis

Two Sample Behrens-Fisher problem for functional data
The problem and test statistics
Null distribution approximation and asymptotic power
Simulation and Real data analysis

Concluding Remarks and future work
Functional data

- Functional linear models:

\[ y_i(t) = x_i^T \beta(t) + v_i(t), v_i(t) \overset{i.i.d.}{\sim} \text{RP}(0, \gamma), \]

where \( t \in \mathcal{T} = [a, b], i = 1, \ldots, n. \)

- By the Karhunen-Loève expansion,

\[ v_i(t) = \sum_{r=1}^{m} \xi_{ir} \phi_r(t), r = 1, \ldots, m, \quad (1) \]

where \( \xi_{ir} \) are independent random variables with mean 0 and variance \( \lambda_r \), \( \phi_r(t) \) are orthonormal and smoothed basis functions on \( \mathcal{T} \) and \( m \) stands for the largest number satisfying \( \lambda_r > 0 \) for \( r = 1, 2, \ldots, m. \)
Functional data

- Functional linear models:

\[ y_i(t) = x_i^T \beta(t) + \nu_i(t), \nu_i(t) \sim \text{RP}(0, \gamma), \]

where \( t \in T = [a, b], i = 1, \ldots, n. \)

- By the Karhunen-Loève expansion,

\[ \nu_i(t) = \sum_{r=1}^{m} \xi_{ir} \phi_r(t), r = 1, \ldots, m, \]

where \( \xi_{ir} \) are independent random variables with mean 0 and variance \( \lambda_r \), \( \phi_r(t) \) are orthonormal and smoothed basis functions on \( T \) and \( m \) stands for the largest number satisfying \( \lambda_r > 0 \) for \( r = 1, 2, \ldots, m. \)
Consequently we have

\[ \gamma(s, t) = \text{Cov}(v_i(s), v_i(t)) = \sum_{r=1}^{m} \lambda_r \phi_r(s) \phi_r(t). \quad (2) \]

Here \( \lambda_r \) and \( \phi_r(t), r = 1, 2, \ldots, m \), are usually known as the eigenvalue and eigenfunction of \( \gamma(s, t) \), and "orthonormal" means \( \phi_r(t) \) satisfies

\[ \int_{\mathcal{T}} \phi_{r_1}(t) \phi_{r_2}(t) dt = \begin{cases} 1, & \text{when } r_1 = r_2, \\ 0, & \text{when } r_1 \neq r_2. \end{cases} \]
For further using, we define

\[ \text{tr}(\gamma) = \int_{\mathcal{T}} \gamma(t, t) dt, \quad \text{and} \quad \gamma \otimes^2 = \int_{\mathcal{T}} \gamma(s, u) \gamma(u, t) du. \]

Then we have

\[ \text{tr}(\gamma) = \sum_{r=1}^{m} \lambda_r, \quad \text{and} \quad \text{tr}(\gamma \otimes^2) = \sum_{r=1}^{m} \lambda_r^2. \]

We assume \( \text{tr}(\gamma) < \infty. \)
Observed data with noise:

\[ y_i(t_{ij}) = x_i^T \beta(t_{ij}) + v_i(t_{ij}) + \epsilon_i(t_{ij}), \quad j = 1, \ldots, n_i, \]

Reconstruction of observed data: We can reconstruct the individual functions from a discrete functional data set using local polynomial smoothing and the effects of substitutions of the individual functions with their reconstructions can be ignored asymptotically (Zhang and Chen (2007)).
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Faraway (1997) pointed out the inappropriateness of traditional multivariate test statistics and proposed a bootstrap-based testing method for comparing two nested models.

Shen and Faraway (2004) proposed an $F$-type test for the same nested models in Faraway (1997) and they approximated the null distribution of the test statistic by a usual $F$ random variable. They also compared the $F$-type test with log-likelihood ratio test and demonstrated that the latter test is less powerful.
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Zhang and Chen (2007) studied the general linear hypothesis testing problem for functional linear models and proposed an $L^2$-norm test.

Ramsay and Silverman (2005) proposed a pointwise $F$-test for functional one-way ANOVA model when they analyzed the Canadian climate data. This pointwise $F$-test motivates our Globalized pointwise $F$-test (GPF).
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Cuevas, Febrero, and Fraiman (2004) also studied one-way ANOVA for functional data. They proposed an $F$-type test and adopted a bootstrap method to approximate the null distribution. When sample size is large, this bootstrap approach is not preferable.
Cuesta-Albertos and Febrero (2010) proposed a random-projection based testing procedure to handle a two-way ANOVA problem for functional data.

The key idea of their method is to project a functional data set onto $k$ randomly-generated directions separately so that the associated two-way ANOVA problem reduces to $k$ univariate two-way ANOVA problems.
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This approach is easy to understand and fast to compute and it may be extended to higher-way ANOVA for functional data in a straightforward way.

When the original functional data are not Gaussian and/or not homogeneous, the resulting univariate data are also not normal and/or homogeneous.

Another difficulty is how to summarize the \( k \) two-way ANOVA results since the \( k \) results may not be consistent.

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Our testing procedures are based on this framework:

- Propose the test statistic and derive the asymptotic expression.
- Find the null distribution by a Welch-Satterthwaite $\chi^2$ approximation. (Satterthwaite 1941, Welch 1947, Zhang 2005)
- Derive the asymptotic power of the test statistics.
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Concluding Remarks and future work
One-way ANOVA problem

- The $k$-sample model:
  \[
  y_{i1}(t), \ldots, y_{in_i}(t) \sim \text{GP}(\mu_i, \gamma), \quad t \in \mathcal{T}, \quad i = 1, \ldots, k, \tag{3}
  \]
  where $\text{GP}(\mu, \gamma)$ denotes a Gaussian process with mean function $\mu(t)$ and covariance function $\gamma(s, t)$.

- It is often interesting to test:
  \[
  H_0 : \mu_1(t) = \mu_2(t) = \cdots = \mu_k(t), \quad t \in \mathcal{T}, \tag{4}
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The $k$ mean functions are often decomposed as 
\[ \mu_i(t) = \mu_0(t) + \alpha_i(t), \ i = 1, 2, \cdots, k, \]
where $\mu_0(t)$ is the grand mean function and $\alpha_i(t), \ i = 1, 2, \cdots, k$ are the $k$ main-effect functions so that (4) is often equivalently written as the problem for testing the equality of the main-effect functions:

\[ H_0 : \alpha_1(t) = \alpha_2(t) = \cdots = \alpha_k(t), \ t \in T. \quad (5) \]
Pointwise $F$ test

The pointwise $F$-test (Ramsay and Silverman (2005)):

$$F_n(t) = \frac{\text{SSR}_n(t)/(k - 1)}{\text{SSE}_n(t)/(n - k)},$$

(6)

where $\text{SSR}_n(t) = \sum_{i=1}^{k} n_i [\bar{y}_i(t) - \bar{y}_i. (t)]^2$ and

$\text{SSE}_n(t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} [y_{ij}(t) - \bar{y}_i(t)]^2$. 
Globalize the pointwise $F$ test

- **Advantage:** for any given $t \in \mathcal{T}$, $F_n(t) \sim F_{k-1, n-k}$.
- **Disadvantage:** time-consuming to test on each point; difficult to summarize the results.
- This motivates the GPF test in the following way:

$$T_n = \int_{\mathcal{T}} F_n(t) \, dt.$$ 

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- This motivates the GPF test in the following way:

$$T_n = \int_{\mathcal{T}} F_n(t) \, dt.$$  \hspace{1cm} (7)
SSR$_n(t)$ can be expressed as:

$$SSR_n(t) = z(t)^T(I_k - b_nb_n^T/n)z(t),$$

where $z(t) = D^{1/2}[\bar{y}_1(t), \bar{y}_2(t), \cdots, \bar{y}_k(t)]^T,$ $b_n = [n_1^{1/2}, n_2^{1/2}, \cdots, n_k^{1/2}]^T,$ and $D = \text{diag}(n_1, n_2, \cdots, n_k).$

$z(t) \sim \text{GP}(\mu_z, \gamma I_k),$ where $\mu_z(t) = D^{1/2} \mu(t)$ with $\mu(t) = [\mu_1(t), \mu_2(t), \cdots, \mu_k(t)]^T.$ $I_k - b_nb_n^T/n$ is an idempotent matrix with rank $k - 1,$ then there exists an orthonormal matrix $P:

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\[
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\]
Set $u(t) = Pz(t)$ so that we have

$$u(t) = [u_1(t), u_2(t), \ldots, u_k(t)]^T \sim \text{GP}(P\mu_z, \gamma I_k).$$

(9)

Thus, $u_1(t), u_2(t), \ldots, u_k(t)$ are independent Gaussian processes, each having the covariance function $\gamma(s, t)$. We then have

$$\text{SSR}_n(t) \overset{d}{=} \sum_{i=1}^{k-1} u_i^2(t),$$

(10)
An unbiased estimator of $\gamma(s, t)$ is given by

$$\hat{\gamma}(s, t) = (n - k)^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_i(s)][y_{ij}(t) - \bar{y}_i(t)].$$

It implies that $\text{SSE}_n(t)/(n - k) = \hat{\gamma}(t, t)$.

By the law of large numbers, for any given $t \in \mathcal{T}$, as $n \to \infty$, we have

$$\hat{\gamma}(t, t) = \text{SSE}_n(t)/(n - k) \xrightarrow{a.s.} \gamma(t, t). \quad (11)$$
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It implies that $\text{SSE}_n(t)/(n - k) = \hat{\gamma}(t, t)$.

By the law of large numbers, for any given $t \in T$, as $n \to \infty$, we have

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\hat{\gamma}(t, t) = \text{SSE}_n(t)/(n - k) \overset{a.s.}{\to} \gamma(t, t).
$$

(11)
Some notations

- \( w(t) = [w_1(t), w_2(t), \cdots, w_{k-1}(t)]^T \) where \( w_i(t) = u_i(t) / \sqrt{\gamma(t, t)} \), \( i = 1, 2, \cdots, k - 1 \).

- \( \mu_w(t) = Ew(t) = \frac{(I_{k-1,0})P\mu_z(t)}{\sqrt{\gamma(t, t)}} \).

- \( \Gamma_w(s, t) = \text{Cov}(w(s), w(t)) = \gamma_w(s, t)I_{k-1} \), where \( \gamma_w(s, t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}} \).
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Some notations

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Here are some properties:

- Since $u(t)$ is Gaussian distributed, then almost surely, $u(t)$ is continuous on $\mathcal{T}$.
- For any $t$, $\gamma(t, t) > 0$ and it is continuous on $\mathcal{T}$. 
Here are some properties:

- Since \( u(t) \) is Gaussian distributed, then almost surely, \( u(t) \) is continuous on \( \mathcal{T} \).
- For any \( t \), \( \gamma(t, t) > 0 \) and it is continuous on \( \mathcal{T} \).
Asymptotic expression of $T_n$

**Theorem**

Assume the $k$ functional samples satisfy (3) with $\gamma(t, t) > 0$ for all $t \in \mathcal{T}$ and that $\sup_{s,t} |\hat{\gamma}(s, t) - \gamma(s, t)| = o_p(1)$, then as $n \to \infty$, we have

$$T_n \overset{d}{=} (k - 1)^{-1} \int_{\mathcal{T}} \|w(t)\|^2 dt + o_p(1). \quad (12)$$
Theorem

Under the conditions of above theorem, as $n \to \infty$, we have

$$T_n \overset{d}{=} (k - 1)^{-1} \left[ \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \pi_r^2 \right] + o_p(1), \quad (13)$$

where $A_r \sim \chi^2_{k-1}(\lambda_r^{-1} \pi_r^2)$, $r = 1, 2, \cdots, m$, and

$$\pi_r^2 = \| \int_{\mathcal{T}} \mu_w(t) \phi_r(t) dt \|^2, \quad r = 1, 2, \cdots, \infty.$$
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Concluding Remarks and future work
Approximating the Null Distribution

- Under the null hypothesis, $T_n$ is asymptotically a $\chi^2$-mixture, thus we adopt the Welch-Satterthwaite approximation to the null distribution.

- The key idea is to approximate the null distribution of $T_n$ by that of a $\chi^2$ random variable multiplied by a constant, namely, $R_w \overset{d}{=} \beta_w \chi^2_{d_w}$ via matching the means and variances of $T_n$ and $R_w$ to determine the parameters $\beta_w$ and $d_w$. 
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Theorem

Under the null hypothesis, as $n \to \infty$, we have

$$E(T_n) = \frac{n - k}{n - k - 2} (b - a) = (b - a) + o(1),$$

$$\text{Var}(T_n) = \frac{2 \text{tr}(\gamma_w^\otimes 2)}{k - 1} + o(1).$$
Compare to the first two moments of $R_w$ we get

$$\beta_w = \frac{(n - k - 2)\text{tr}(\gamma_w \otimes^2)}{(k - 1)(n - k)(b - a)}, \quad d_w = \frac{(k - 1)(n - k)^2(b - a)^2}{(n - k - 2)^2\text{tr}(\gamma_w \otimes^2)},$$

ignoring the higher order terms. It is easy to see that as $n \to \infty$, we have

$$\beta_w \to \beta^*_w = \frac{\text{tr}(\gamma_w \otimes^2)}{(k - 1)(b - a)}, \quad d_w \to d^*_w = \frac{(k - 1)(b - a)^2}{\text{tr}(\gamma_w \otimes^2)}.$$
In practice, the parameters $\beta_w$ and $d_w$ have to be estimated based on the data. Their natural estimators

$$\hat{\beta}_w = \frac{(n - k - 2)\text{tr}(\hat{\gamma}^\otimes 2)}{(k - 1)(n - k)(b - a)}, \quad \hat{d}_w = \frac{(k - 1)(n - k)^2(b - a)^2}{(n - k - 2)^2\text{tr}(\hat{\gamma}^\otimes 2)},$$

are obtained via replacing the covariance function $\gamma_w(s, t)$ by its estimator

$$\hat{\gamma}_w(s, t) = \frac{\hat{\gamma}(s, t)}{\sqrt{\hat{\gamma}(s, s)\hat{\gamma}(t, t)}}.$$
For any given significance level $\alpha$, the oracle and estimated critical value of $T_n$ is specified as

$$T^* = \beta^* \chi^2_{d^*}(\alpha), \quad \hat{T}_{n,\alpha} = \hat{\beta}_w \chi^2_{d_w}(\alpha),$$

where $\chi^2_{\nu}(\alpha)$ denotes the upper $100\alpha$ percentile of $\chi^2_{\nu}$ for any $\nu > 0$.

**Theorem**

As $n \to \infty$, we have

$$\hat{\beta}_w \xrightarrow{a.s.} \beta^*_w, \quad \hat{d}_w \xrightarrow{a.s.} d^*_w, \quad \text{and} \quad \hat{T}_{n,\alpha} \xrightarrow{a.s.} T^*_\alpha. \quad (14)$$
For any given significance level $\alpha$, the oracle and estimated critical value of $T_n$ is specified as

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(14)
Asymptotic power

- Suppose we have a sequence of alternatives:

\[ H_{1n} : \mu_i(t) = \mu_0(t) + n^{-\tau/2}d_i(t), \; i = 1, 2, \cdots, k, \tag{15} \]

where \( \tau \) is some constant with \( 0 < \tau < 1 \), and \( d(t) = [d_1(t), \cdots, d_k(t)]^T \) is any fixed real vector of functions, independent of \( n \).

- Suppose \( \lim_{n \to \infty} D/n = \tilde{D} \) where \( \tilde{D} \) is some nonsingular matrix.

- **Theorem**

  As \( n \to \infty \), the asymptotic power of \( T_n \)

  \[ P(T_n \geq \hat{T}_{n,\alpha}|H_{1n}) \to 1. \]
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\[ H_{1n} : \mu_i(t) = \mu_0(t) + n^{-\tau/2}d_i(t), \quad i = 1, 2, \cdots, k, \quad (15) \]

where \( \tau \) is some constant with \( 0 < \tau < 1 \), and \( d(t) = [d_1(t), \cdots, d_k(t)]^T \) is any fixed real vector of functions, independent of \( n \).

- Suppose \( \lim_{n \to \infty} D/n = \tilde{D} \) where \( \tilde{D} \) is some nonsingular matrix.

- Theorem

As \( n \to \infty \), the asymptotic power of \( T_n \)

\[ P(T_n \geq \hat{T}_{n,\alpha}|H_{1n}) \to 1. \]
Asymptotic power

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▶ Theorem

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Outline

Introduction

A GPF test for one-way ANOVA problem
- The one-way problem and GPF test
- Null distribution approximation and asymptotic power
- Simulation and Real data analysis

The $L^2$-norm and $F$-type test for two-way ANOVA Problem
- $L^2$-norm test under non-Gaussian assumption
- $F$-type test under Gaussian assumption
- Simulation and Real data analysis

Two Sample Behrens-Fisher problem for functional data
- The problem and test statistics
- Null distribution approximation and asymptotic power
- Simulation and Real data analysis

Concluding Remarks and future work

Liang Xuehua
On Some Hypothesis Testing Problems in Functional Data Analysis
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We conduct a simulation study to compare the performance of GPF test with two other tests.

The $L^2$-norm based test (Faraway 1997; Zhang and Chen 2007) under one-way ANOVA problem is given as

$$S_n = \int_\mathcal{T} \text{SSR}_n(t) dt.$$ 

The $F$-type test (Shen and Faraway 2004; Zhang 2011) is given as

$$F_n = \frac{\int_\mathcal{T} \text{SSR}_n(t) dt / (k - 1)}{\int_\mathcal{T} \text{SSE}_n(t) dt / (n - k)}.$$
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The $k$ functional samples were generated from the model

$$y_{ij}(t) = \mu_i(t) + \nu_{ij}(t)$$

where

$$\mu_i(t) = c_i^T [1, t, t^2, t^3]^T, \quad \nu_{ij}(t) = b_{ij}^T \psi(t), \quad t \in T,$$

$$b_{ij} = [b_{ij1}, \cdots, b_{ijq}]^T \overset{i.i.d.}{\sim} N_q(0, \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_q)),$$

$$j = 1, 2, \cdots, n_i; \ i = 1, 2, \cdots, k,$$

- $c_1$ is a constant vector and $c_i = c_1 + (i - 1) \delta u, \ i = 2, \cdots, k$
- $\lambda_r = a \rho^{r-1}$, for some $a > 0$ and $0 < \rho < 1$.
- Basis functions are $\psi_1(t) = 1, \ \psi_{2r}(t) = \sqrt{2} \sin(2\pi rt), \ \psi_{2r+1}(t) = \sqrt{2} \cos(2\pi rt), \ t \in [0, 1], \ r = 1, \cdots, (q - 1)/2.$
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$$
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$$
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$$
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$$
\begin{align*}
\mu_i(t) &= c_i^T [1, t, t^2, t^3]^T, \\
\nu_{ij}(t) &= b_{ij}^T \Psi(t), \\
b_{ij} &= [b_{ij1}, \ldots, b_{ijq}]^T \sim \text{i.i.d. } \mathcal{N}_q(0, \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_q)),
\end{align*}
$$

$j = 1, 2, \ldots, n_i; i = 1, 2, \ldots, k,$

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\end{align*}
\]
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\end{align*}
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\begin{itemize}
\item $c_1$ is a constant vector and $c_i = c_1 + (i - 1)\delta u$, $i = 2, \ldots, k$
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\end{itemize}
### Table: Empirical sizes and powers.

<table>
<thead>
<tr>
<th>ρ</th>
<th>([n_1, n_2, n_3])</th>
<th>Method</th>
<th>(δ = 0)</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>([15,15,15])</td>
<td>(L^2)-norm</td>
<td>0.060</td>
<td>0.175</td>
<td>0.622</td>
<td>0.956</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F)-type</td>
<td>0.049</td>
<td>0.156</td>
<td>0.584</td>
<td>0.943</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPF</td>
<td>0.059</td>
<td>0.173</td>
<td>0.618</td>
<td>0.955</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>([30,20,25])</td>
<td>(L^2)-norm</td>
<td>0.053</td>
<td>0.299</td>
<td>0.900</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F)-type</td>
<td>0.048</td>
<td>0.280</td>
<td>0.888</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
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<td></td>
<td></td>
<td>GPF</td>
<td>0.052</td>
<td>0.299</td>
<td>0.900</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
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<tr>
<td></td>
<td>([50,30,40])</td>
<td>(L^2)-norm</td>
<td>0.055</td>
<td>0.451</td>
<td>0.995</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
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<td></td>
<td></td>
<td>(F)-type</td>
<td>0.054</td>
<td>0.440</td>
<td>0.994</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPF</td>
<td>0.055</td>
<td>0.452</td>
<td>0.995</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.50</td>
<td>([15,15,15])</td>
<td>(L^2)-norm</td>
<td>0.057</td>
<td>0.078</td>
<td>0.143</td>
<td>0.272</td>
<td>0.448</td>
<td>0.670</td>
</tr>
<tr>
<td></td>
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<td>(F)-type</td>
<td>0.051</td>
<td>0.069</td>
<td>0.129</td>
<td>0.247</td>
<td>0.426</td>
<td>0.641</td>
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<tr>
<td></td>
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<td>GPF</td>
<td>0.064</td>
<td>0.085</td>
<td>0.153</td>
<td>0.284</td>
<td>0.470</td>
<td>0.690</td>
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<td></td>
<td>([30,20,25])</td>
<td>(L^2)-norm</td>
<td>0.059</td>
<td>0.087</td>
<td>0.224</td>
<td>0.467</td>
<td>0.742</td>
<td>0.930</td>
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<td>(F)-type</td>
<td>0.055</td>
<td>0.082</td>
<td>0.212</td>
<td>0.451</td>
<td>0.731</td>
<td>0.924</td>
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<tr>
<td></td>
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<td>GPF</td>
<td>0.064</td>
<td>0.094</td>
<td>0.234</td>
<td>0.485</td>
<td>0.759</td>
<td>0.936</td>
</tr>
<tr>
<td></td>
<td>([50,30,40])</td>
<td>(L^2)-norm</td>
<td>0.050</td>
<td>0.119</td>
<td>0.349</td>
<td>0.717</td>
<td>0.945</td>
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<td></td>
<td></td>
<td>(F)-type</td>
<td>0.048</td>
<td>0.113</td>
<td>0.340</td>
<td>0.707</td>
<td>0.943</td>
<td>0.996</td>
</tr>
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<td>GPF</td>
<td>0.053</td>
<td>0.122</td>
<td>0.361</td>
<td>0.725</td>
<td>0.951</td>
<td>0.997</td>
</tr>
<tr>
<td>0.90</td>
<td>([15,15,15])</td>
<td>(L^2)-norm</td>
<td>0.039</td>
<td>0.048</td>
<td>0.061</td>
<td>0.086</td>
<td>0.136</td>
<td>0.197</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F)-type</td>
<td>0.033</td>
<td>0.044</td>
<td>0.054</td>
<td>0.077</td>
<td>0.124</td>
<td>0.181</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPF</td>
<td>0.051</td>
<td>0.060</td>
<td>0.075</td>
<td>0.102</td>
<td>0.159</td>
<td>0.219</td>
</tr>
<tr>
<td></td>
<td>([30,20,25])</td>
<td>(L^2)-norm</td>
<td>0.047</td>
<td>0.057</td>
<td>0.076</td>
<td>0.151</td>
<td>0.251</td>
<td>0.420</td>
</tr>
<tr>
<td></td>
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<td>(F)-type</td>
<td>0.043</td>
<td>0.053</td>
<td>0.071</td>
<td>0.143</td>
<td>0.242</td>
<td>0.407</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPF</td>
<td>0.054</td>
<td>0.063</td>
<td>0.085</td>
<td>0.165</td>
<td>0.267</td>
<td>0.438</td>
</tr>
<tr>
<td></td>
<td>([50,30,40])</td>
<td>(L^2)-norm</td>
<td>0.046</td>
<td>0.064</td>
<td>0.127</td>
<td>0.241</td>
<td>0.445</td>
<td>0.676</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F)-type</td>
<td>0.044</td>
<td>0.062</td>
<td>0.122</td>
<td>0.235</td>
<td>0.436</td>
<td>0.670</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPF</td>
<td>0.056</td>
<td>0.070</td>
<td>0.135</td>
<td>0.251</td>
<td>0.458</td>
<td>0.683</td>
</tr>
</tbody>
</table>
Real data application

- The Canadian temperature data: the daily temperature records of 35 Canadian weather-stations over 365 days.
- 15 are in Eastern, another 15 in Western and the remaining 5 in Northern Canada.
- We aim to test the difference of mean temperature of the three regions during a year.
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Table: One-way ANOVA of the Canadian temperature data by the $L^2$-norm based test, the $F$-type test and the GPF test. The quantities were computed with the resolution $M = 1000$ where and throughout, “8.6e4” denotes “8.6 × 10^4”.

<table>
<thead>
<tr>
<th>Period</th>
<th>$L^2$-norm Based test</th>
<th>$F$-type test</th>
<th>GPF test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S_n$</td>
<td>$F_n$</td>
<td>$T_n$</td>
</tr>
<tr>
<td></td>
<td>$P$-value</td>
<td>$P$-value</td>
<td>$P$-value</td>
</tr>
<tr>
<td>Spring</td>
<td>$8.6e4$</td>
<td>$17.3$</td>
<td>$16.7$</td>
</tr>
<tr>
<td></td>
<td>$1.7e-9$</td>
<td>$1.2e-6$</td>
<td>$1.4e-9$</td>
</tr>
<tr>
<td>Summer</td>
<td>$1.8e4$</td>
<td>$10.9$</td>
<td>$11.5$</td>
</tr>
<tr>
<td></td>
<td>$1.0e-5$</td>
<td>$1.6e-4$</td>
<td>$3.5e-6$</td>
</tr>
<tr>
<td>Autumn</td>
<td>$7.6e4$</td>
<td>$28.3$</td>
<td>$27.7$</td>
</tr>
<tr>
<td></td>
<td>$5.4e-15$</td>
<td>$7.0e-9$</td>
<td>$2.0e-15$</td>
</tr>
<tr>
<td>Winter</td>
<td>$1.2e5$</td>
<td>$13.3$</td>
<td>$13.6$</td>
</tr>
<tr>
<td></td>
<td>$1.3e-6$</td>
<td>$5.1e-5$</td>
<td>$5.7e-7$</td>
</tr>
<tr>
<td>Year</td>
<td>$3.1e5$</td>
<td>$16.3$</td>
<td>$17.4$</td>
</tr>
<tr>
<td></td>
<td>$1.9e-10$</td>
<td>$2.4e-7$</td>
<td>$6.8e-13$</td>
</tr>
</tbody>
</table>
Table: *P*-values of the $L^2$-norm based test, the $F$-type test and the GPF test for testing the equality of the groupwise mean temperature functions for the Canadian temperature data, calculated with the resolution $M = 1000$.

<table>
<thead>
<tr>
<th>Period</th>
<th>East vs West</th>
<th>East vs North</th>
<th>West vs North</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GPF</td>
<td>GPF</td>
<td>GPF</td>
</tr>
<tr>
<td>Spring</td>
<td>0.87</td>
<td>2.4e−11</td>
<td>4.7e−7</td>
</tr>
<tr>
<td>Summer</td>
<td>0.08</td>
<td>4.9e−6</td>
<td>4.5e−4</td>
</tr>
<tr>
<td>Autumn</td>
<td>0.02</td>
<td>0</td>
<td>4.1e−9</td>
</tr>
<tr>
<td>Winter</td>
<td>0.30</td>
<td>0</td>
<td>9.8e−5</td>
</tr>
<tr>
<td>Year</td>
<td>0.11</td>
<td>0</td>
<td>3.2e−8</td>
</tr>
</tbody>
</table>
Two-Way ANOVA Problem

- A two-way design is given as:

\[ y_{ijk}(t) = \mu_{ij}(t) + \nu_{ijk}(t), \quad \nu_{ijk}(t) \sim \text{RP}(\mu_{ij}, \gamma), \quad (16) \]

where \( i = 1, \cdots, a, \ j = 1, \cdots, b, \ k = 1, \cdots, n_{ij}, \ t \in \mathcal{T}. \)

- Cell mean \( \mu_{ij}(t) \) is often decomposed as

\[ \mu_{ij}(t) = \eta(t) + \alpha_i(t) + \beta_j(t) + \theta_{ij}(t). \quad (17) \]
Two-Way ANOVA Problem

- A two-way design is given as:

\[ y_{ijk}(t) = \mu_{ij}(t) + v_{ijk}(t), \quad v_{ijk}(t) \sim \text{RP} (\mu_{ij}, \gamma), \quad \text{(16)} \]

where \( i = 1, \cdots, a, \; j = 1, \cdots, b, \; k = 1, \cdots, n_{ij}, \; t \in \mathcal{T} \).

- Cell mean \( \mu_{ij}(t) \) is often decomposed as

\[ \mu_{ij}(t) = \eta(t) + \alpha_i(t) + \beta_j(t) + \theta_{ij}(t). \quad \text{(17)} \]
A UV-decomposition: when \( u = [u_1, \cdots, u_a] \) and 
\( v = [v_1, \cdots, v_b] \) satisfy \( \sum_{i=1}^{a} u_i = 1, \sum_{j=1}^{b} v_j = 1 \), we can identify

\[
\eta(t) = \sum_{i=1}^{a} \sum_{j=1}^{b} u_i v_j \mu_{ij}(t),
\]

\[
\alpha_i(t) = \sum_{j=1}^{b} v_j \mu_{ij}(t) - \eta(t),
\]

\[
\beta_j(t) = \sum_{i=1}^{a} u_i \mu_{ij}(t) - \eta(t),
\]

\[
\theta_{ij}(t) = \mu_{ij}(t) - \alpha_i(t) - \beta_j(t) - \eta(t).
\]
The above expressions can be re-written as

\[ \alpha = A_\alpha \mu(t), \ \beta = A_\beta \mu(t), \ \theta = A_{\alpha\beta} \mu(t), \]

where \( \mu(t) = [\mu_{11}(t), \ldots, \mu_{1b}(t), \ldots, \mu_{a1}(t), \ldots, \mu_{ab}(t)]^T \),

\( A_\alpha = (I_a - 1_a u^T) \otimes v^T \), \( A_\beta = u^T \otimes (I_b - 1_b v^T) \), and

\( A_{\alpha\beta} = (I_a - 1_a u^T) \otimes (I_b - 1_b v^T) \).

 Equal weight system

\[ u_i = 1/a, \ v_j = 1/b \]

 Size-adjusted system

\[ u_i = \sum_{j=1}^{b} n_{ij}/N, \ v_j = \sum_{i=1}^{a} n_{ij}/N. \]
The above expressions can be re-written as

\[ \alpha = A_\alpha \mu(t), \; \beta = A_\beta \mu(t), \; \theta = A_{\alpha\beta} \mu(t), \]

where \( \mu(t) = [\mu_{11}(t), \cdots, \mu_{1b}(t), \cdots, \mu_{a1}(t), \cdots, \mu_{ab}(t)]^T \),
\( A_\alpha = (I_a - 1_a u^T) \otimes v^T \), \( A_\beta = u^T \otimes (I_b - 1_b v^T) \), and
\( A_{\alpha\beta} = (I_a - 1_a u^T) \otimes (I_b - 1_b v^T) \).

Equal weight system

\[ u_i = 1/a, \; v_j = 1/b \]

Size-adjusted system

\[ u_i = \sum_{j=1}^{b} n_{ij}/N, \; v_j = \sum_{i=1}^{a} n_{ij}/N. \]
The above expressions can be re-written as

\[
\alpha = A_\alpha \mu(t), \quad \beta = A_\beta \mu(t), \quad \theta = A_{\alpha\beta} \mu(t),
\]

where \( \mu(t) = [\mu_{11}(t), \ldots, \mu_{1b}(t), \ldots, \mu_{a1}(t), \ldots, \mu_{ab}(t)]^T \),

\[
A_\alpha = (I_a - 1_a u^T) \otimes v^T, \quad A_\beta = u^T \otimes (I_b - 1_b v^T), \quad \text{and} \quad A_{\alpha\beta} = (I_a - 1_a u^T) \otimes (I_b - 1_b v^T).
\]

- **Equal weight system**

\[
u_i = 1/a, \quad v_j = 1/b
\]

- **Size-adjusted system**

\[
u_i = \sum_{j=1}^{b} n_{ij}/N, \quad v_j = \sum_{i=1}^{a} n_{ij}/N.
\]
What of interest is to test the following null hypotheses:

\[ H_A : \alpha_1(t) = \alpha_2(t) = \cdots = \alpha_a(t) = 0, \]
\[ H_B : \beta_1(t) = \beta_2(t) = \cdots = \beta_b(t) = 0, \quad (18) \]
\[ H_{A\ast B} : \theta_{11}(t) = \theta_{12}(t) = \cdots = \theta_{ab}(t) = 0. \]

\[ H_A : H_{\alpha\alpha}(t) = 0, \]
\[ H_B : H_{\beta\beta}(t) = 0, \quad (19) \]
\[ H_{A\ast B} : H_{\alpha\beta\theta}(t) = 0, \]
What of interest is to test the following null hypotheses:

\[ \begin{align*}
H_A & : \quad \alpha_1(t) = \alpha_2(t) = \cdots = \alpha_a(t) = 0, \\
H_B & : \quad \beta_1(t) = \beta_2(t) = \cdots = \beta_b(t) = 0, \\
H_{A*}\!B & : \quad \theta_{11}(t) = \theta_{12}(t) = \cdots = \theta_{ab}(t) = 0. 
\end{align*} \quad (18) \]

\[ \begin{align*}
H_A & : \quad H_\alpha \alpha(t) = 0, \\
H_B & : \quad H_\beta \beta(t) = 0, \\
H_{A*}\!B & : \quad H_\alpha \beta \theta(t) = 0. 
\end{align*} \quad (19) \]
Then the hypothesis testing problems $H_A$, $H_B$ and $H_{A\times B}$ are equivalent to

$$
H_A : \quad C_\alpha \mu(t) = 0, \quad \text{where} \quad C_\alpha = H_\alpha A_\alpha,
$$

$$
H_B : \quad C_\beta \mu(t) = 0, \quad \text{where} \quad C_\beta = H_\beta A_\beta,
$$

$$
H_{A\times B} : \quad C_{\alpha\beta} \mu(t) = 0, \quad \text{where} \quad C_{\alpha\beta} = H_{\alpha\beta} A_{\alpha\beta}.
$$

We can further simplify the matrices $C_\alpha$, $C_\beta$, and $C_{\alpha\beta}$ as

$$
C_\alpha = (I_{a-1}, -1_{a-1}) \otimes v^T,
$$

$$
C_\beta = u^T \otimes (I_{b-1}, -1_{b-1}),
$$

$$
C_{\alpha\beta} = (I_{a-1}, -1_{a-1}) \otimes (I_{b-1}, -1_{b-1}).
$$
General linear hypothesis testing (GLHT) problem:

\[ H_0 : \mathbf{C} \mu(t) = 0 \text{ vs } H_1 : \mathbf{C} \mu(t) \neq 0. \]  (22)

In fact, when \( \mathbf{C} = \mathbf{C}_\alpha, \mathbf{C}_\beta \) and \( \mathbf{C}_{\alpha\beta} \), the above GLHT problem reduces to \( H_A, H_B, \) and \( H_{A*B} \). We use \( q \) to denote the rank of \( \mathbf{C} \) which will be \( a - 1, b - 1, \) and \( (a - 1)(b - 1) \) respectively.
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$L^2$-norm test on Non-Gaussian assumption

Under non-Gaussian assumption, the response functions are

$$y_{ijk}(t) = \mu_{ij}(t) + \nu_{ijk}(t), \quad \nu_{ijk}(t) \sim \text{RP}(0, \gamma).$$  \hspace{1cm} (23)

We want to test the GLHT problem (22). The $L^2$-norm test statistic is defined as

$$T = \int_{\mathcal{T}} (C\hat{\mu}(t))^T [CDC^T]^{-1} (C\hat{\mu}(t)) dt.$$  \hspace{1cm} (24)
Theorem
As $N \to \infty$, we have

$$T \overset{d}{=} \int_T \|\omega(t)\|^2 dt = \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \pi_r^2 + o_p(1),$$

where $\omega(t) \sim GP(\mu_\omega(t), I_q \gamma)$, $A_r \sim \chi_q^2(\lambda_r^{-1}\pi_r^2)$,

$$\pi_r^2 = \| \int_T \mu_\omega(t) \phi_r(t) dt \|^2, \quad r = 1, 2, \ldots, m,$$

and $\mu_\omega(t) = (CDC^T)^{-1/2} C \mu(t)$.

The asymptotic expression of $T$ under the null hypothesis is

$$T \overset{d}{=} T^* + o_p(1) = \sum_{r=1}^{m} \lambda_r A_r + o_p(1), \quad A_r \overset{i.i.d.}{\sim} \chi_q^2. \quad (25)$$
The approximate null distribution of $T$ is already found in the previously. By the Welch-Satterthwaite $\chi^2$-approximation, we have $T^* \approx \frac{c}{\chi^2_d}$, where

$$c = \frac{\text{tr}(\gamma \otimes 2)}{\text{tr}(\gamma)}, \quad d = q\kappa, \quad \kappa = \frac{\text{tr}^2(\gamma)}{\text{tr}(\gamma \otimes 2)}.$$

An unbiased estimator of covariance function is

$$\hat{\gamma}(s, t) = (N - ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} [y_{ijk}(s) - \bar{y}_{ij}(s)][y_{ijk}(t) - \bar{y}_{ij}(t)].$$
The approximate null distribution of $T$ is already found in the previously. By the Welch-Satterthwaite $\chi^2$-approximation, we have $T^* \approx c \chi_{d}^2$, where

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The estimators of $c$ and $d$ are also given as

$$
\hat{c} = \frac{\text{tr}(\hat{\gamma} \otimes 2)}{\text{tr}(\hat{\gamma})}, \quad \hat{d} = q\hat{\kappa}, \quad \hat{\kappa} = \frac{\text{tr}^2(\hat{\gamma})}{\text{tr}(\hat{\gamma} \otimes 2)}.
$$

The estimated critical value on level $\alpha$ is $\hat{T}(\alpha) = \hat{c}\chi_{d,\alpha}^2,$

where $\chi_{d,\alpha}^2$ denotes the upper $100\alpha$ percentile of $\chi_d^2.$

Theorem
Assume that $\text{tr}(\gamma) < \infty.$ As $N \to \infty$, we have

$$
\hat{c} \xrightarrow{a.s.} c, \quad \hat{\kappa} \xrightarrow{a.s.} \kappa, \quad \hat{T}(\alpha) \xrightarrow{a.s.} c\chi_{d,\alpha}^2/d,
$$

where $\chi_{d,\alpha}^2$ denotes the upper $100\alpha$ percentile of $\chi_d^2$ with $d = q\kappa.$
The estimators of $c$ and $d$ are also given as

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\hat{c} = \frac{\text{tr}(\hat{\gamma} \otimes 2)}{\text{tr}(\hat{\gamma})}, \quad \hat{d} = q \hat{\kappa}, \quad \hat{\kappa} = \frac{\text{tr}^2(\hat{\gamma})}{\text{tr}(\hat{\gamma} \otimes 2)}.
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The estimated critical value on level $\alpha$ is

$$
\hat{T}(\alpha) = \hat{c} \chi^2_{\hat{d}, \alpha},
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where $\chi^2_{\hat{d}, \alpha}$ denotes the upper $100\alpha$ percentile of $\chi^2_{\hat{d}}$.

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Assume that $\text{tr}(\gamma) < \infty$. As $N \to \infty$, we have

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\hat{c} \xrightarrow{a.s.} c, \quad \hat{\kappa} \xrightarrow{a.s.} \kappa, \quad \hat{T}(\alpha) \xrightarrow{a.s.} c \chi^2_{\hat{d}, \alpha} / d,
$$

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The estimators of $c$ and $d$ are also given as

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\hat{c} = \frac{\text{tr}(\hat{\gamma} \otimes^2)}{\text{tr}(\hat{\gamma})}, \quad \hat{d} = q\hat{\kappa}, \quad \hat{\kappa} = \frac{\text{tr}^2(\hat{\gamma})}{\text{tr}(\hat{\gamma} \otimes^2)}.
\]

The estimated critical value on level $\alpha$ is

\[
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\]

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Assume that $\text{tr}(\gamma) < \infty$. As $N \to \infty$, we have

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*Assume that $\text{tr}(\gamma) < \infty$. As $N \to \infty$, we have*

$$
\hat{c} \xrightarrow{a.s.} c, \quad \hat{\kappa} \xrightarrow{a.s.} \kappa, \quad \hat{T}(\alpha) \xrightarrow{a.s.} c\chi^2_{d,\alpha}/d,
$$

*where $\chi^2_{d,\alpha}$ denotes the upper 100$\alpha$ percentile of $\chi^2_d$ with $d = q\kappa$.*
Asymptotic power

We will study the asymptotic power of the $T$ under a sequence of local alternatives as:

$$H_{1n} : \mathbf{C}\mu(t) = N^{-\tau/2}\mathbf{d}(t).$$  \hspace{1cm} (26)

We also assume as $N \to \infty$, $\frac{n_{ij}}{N} \to \rho_{ij}$, where $0 < \rho_{ij} < 1$.

Theorem

As $n \to \infty$, the asymptotic power of $T$

$$P(T \geq \hat{T}(\alpha)|H_{1n}) \to 1,$$
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We will study the asymptotic power of the $T$ under a sequence of local alternatives as:

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F-type test under Gaussian assumption

- F-type test statistic is

\[
F = \frac{\int_{\mathcal{T}} (\mathbf{C}\hat{\mu}(t))^T \mathbf{CDC}^T \mathbf{C}\hat{\mu}(t) \, dt}{qtr(\hat{\gamma})}.
\]  

(27)

where \( \mathbf{D} = \text{diag}\{n_{11}, \ldots, n_{ab}\} \).
Theorem
Assume that \( \text{tr}(\gamma) < \infty \) and under Gaussian covariance structure, we have

\[
F \overset{d}{=} \frac{\left[ \sum_{r=1}^{m} \lambda_r A_r + \sum_{r=m+1}^{\infty} \pi_r^2 \right]/q}{\sum_{r=1}^{m} \lambda_r B_r / (N - ab)},
\]

where \( A_r, B_r, r = 1, 2, \cdots, m \), are independent random variables, \( A_r \sim \chi^2_q(\lambda_r^{-1} \pi_r^2), B_r \sim \chi^2_{N-ab} \),
\[
\pi_r^2 = \| \int_T \mu_\omega(t) \phi_r(t) \, dt \|^2, \quad r = 1, 2, \cdots, \infty, \text{ and }
\]
\[
\mu_\omega(t) = (CDC^T)^{-1/2} C \mu(t).
\]

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By the above theorem, it is easy to see that under the null hypothesis, as $N \to \infty$, the asymptotic expression of $F$ is

$$F \overset{d}{=} \frac{\sum_{r=1}^{m} \lambda_r A_r / q}{\sum_{r=1}^{m} \lambda_r B_r / (N - ab)} \overset{d}{=} \frac{T^*}{qtr(\gamma)} + o_p(1), \quad (28)$$

where

$$T^* = \sum_{r=1}^{m} \lambda_r A_r, \quad A_r \overset{i.i.d.}{\sim} \chi^2_q. \quad (29)$$
When $m < \infty$, (28) holds because as $N \to \infty$, $rac{B_r}{N^{ab}} \xrightarrow{P} 1$ for each $B_r$.

When $m = \infty$, it is not trivial and we need a proposition.

**Proposition**

Let $\{B^n_r, r, n \in \mathbb{N}\}$ be a triangular array such that for any $n \in \mathbb{N}$, $\{B^n_r, r, n \in \mathbb{N}\}$ are i.i.d $\chi^2_n$ random variables. Then the sequence $S_n = \sum_r \frac{\lambda_r B^n_r}{n}$ converges in distribution to $\Lambda = \sum_r \lambda_r$. 

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When \( m < \infty \), (28) holds because as \( N \to \infty \), \( \frac{B_r}{N^{-ab}} \overset{P}{\to} 1 \) for each \( B_r \).

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**Proposition**

Let \( \{B^r_n, r, n \in \mathbb{N} \} \) be a triangular array such that for any \( n \in \mathbb{N} \), \( \{B^r_n, r, n \in \mathbb{N} \} \) are i.i.d \( \chi^2_n \) random variables. Then the sequence \( S_n = \sum_r \frac{\lambda_r B^r_n}{n} \) converges in distribution to \( \Lambda = \sum_r \lambda_r \).
Null distribution approximation

When the null hypothesis (22) holds, we have

\[ F \overset{d}{=} \frac{\sum_{r=1}^{m} \lambda_r A_r/q}{\sum_{r=1}^{m} \lambda_r B_r/(N - ab)}, \]

where \( A_r \overset{i.i.d.}{\sim} \chi^2_q \) and \( B_r \overset{i.i.d.}{\sim} \chi^2_{N - ab} \).

Note that \( F \) is the ratio of two \( \chi^2 \)-mixtures, thus \( \chi^2 \) approximation can be used.
When the null hypothesis (22) holds, we have

\[ F \overset{d}{=} \frac{\sum_{r=1}^{m} \lambda_r A_r / q}{\sum_{r=1}^{m} \lambda_r B_r / (N - ab)}, \]

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Note that \( F \) is the ratio of two \( \chi^2 \)-mixtures, thus \( \chi^2 \) approximation can be used.
We then have

\[ F \overset{\text{approx.}}{\sim} F_{d_1, d_2}. \]

\[ d_1 = q\kappa, \quad d_2 = (N - ab)\kappa, \quad \kappa = \frac{\text{tr}^2(\gamma)}{\text{tr}(\gamma \otimes 2)}. \]
We then have

\[ F^{\text{approx.}} \sim F_{d_1,d_2}. \]

\[ d_1 = q\kappa, \quad d_2 = (N - ab)\kappa, \quad \kappa = \frac{\text{tr}^2(\gamma)}{\text{tr}(\gamma \otimes 2)}. \] (30)
We need to estimate $\kappa$, or $\gamma(s, t)$. Recall that an unbiased estimator of $\gamma(s, t)$ is

$$
\hat{\gamma}(s, t) = (N - ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} [y_{ijk}(s) - \bar{y}_{ij}(s)][y_{ijk}(t) - \bar{y}_{ij}(t)],
$$

Thus we have the estimators

$$
\hat{d}_1 = q\hat{\kappa}, \quad \hat{d}_2 = (N - ab)\hat{\kappa}, \quad \hat{\kappa} = \frac{\text{tr}^2(\hat{\gamma})}{\text{tr}(\hat{\gamma} \otimes 2)},
$$

so that

$$
F \overset{\text{approx.}}{\sim} F_{\hat{d}_1, \hat{d}_2}. \quad (31)
$$

For a given significance level $\alpha$, the estimated critical value will be $F_{\hat{d}_1, \hat{d}_2}(\alpha)$. 

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F \overset{\text{approx.}}{\sim} F_{\hat{d}_1, \hat{d}_2}. \quad (31)
$$

For a given significance level $\alpha$, the estimated critical value will be $F_{\hat{d}_1, \hat{d}_2}(\alpha)$. 
Theorem

Assume that $\text{tr}(\gamma) < \infty$. As $N \to \infty$, we have

$$\hat{\kappa} \xrightarrow{a.s.} \kappa, \quad \hat{F}_{\hat{d}_1, \hat{d}_2}(\alpha) \xrightarrow{a.s.} \chi^2_{d_1, \alpha}/d_1,$$

where $\chi^2_{d_1, \alpha}$ denotes the upper $100\alpha$ percentile of $\chi^2_{d_1}$ with $d_1 = q\kappa$. 

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Asymptotic power

- We will study the asymptotic power of the $F$ under a sequence of local alternatives as:

$$H_{1n} : C\mu(t) = N^{-\tau/2}d(t).$$

(32)

We also assume as $N \to \infty$, $\frac{n_{ij}}{N} \to \rho_{ij}$, where $0 < \rho_{ij} < 1$.

Denote $\lim_{N \to \infty} ND = \Omega$.

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As $n \to \infty$, the asymptotic power of $F$

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A simulation study

- Firstly, we define the cell mean functions as
  \[ \mu_{ij}(t) = e_{ij}^T [1, t, t^2, t^3]^T, \; i = 1, 2, \ldots, a, \; j = 1, 2, \ldots, b, \]
  where \( e_{ij} = e_0 + ij\delta u, \) \( e_0 \) and \( u \) are constant vectors, and \( \delta \) is a tuning parameter specifying the differences among the cell mean functions \( \mu_{ij}(t). \)

- Secondly, we generated the subject-effect functions \( \nu_{ijk}(t) \) using the Karhunen-Loève expansion (1) so that
  \[ \nu_{ijk}(t) = \sum_{r=1}^{m} \xi_{ijk} \phi_r(t), \; k = 1, 2, \ldots, n_{ij}; \]
  where \( \xi_{ijk} \) are independent random coefficients with \( E(\xi_{ijk}) = 0 \) and \( \text{Var}(\xi_{ijk}) = \lambda_r. \)
A simulation study

- Firstly, we define the cell mean functions as
  \[ \mu_{ij}(t) = e_{ij}^T [1, t, t^2, t^3]^T, \quad i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \]
  where \( e_{ij} = e_0 + ij \delta u \), \( e_0 \) and \( u \) are constant vectors, and \( \delta \) is a tuning parameter specifying the differences among the cell mean functions \( \mu_{ij}(t) \).

- Secondly, we generated the subject-effect functions \( \upsilon_{ijk}(t) \) using the Karhunen-Loève expansion (1) so that
  \[ \upsilon_{ijk}(t) = \sum_{r=1}^{m} \xi_{ijkr} \phi_r(t), \quad k = 1, 2, \ldots, n_{ij}; \]
  where \( \xi_{ijkr} \) are independent random coefficients with \( \mathbb{E}(\xi_{ijkr}) = 0 \) and \( \text{Var}(\xi_{ijkr}) = \lambda_r \).
Basis functions are

\[ \phi_1(t) = 1, \phi_{2r}(t) = \sqrt{2}\sin(2\pi rt), \]
\[ \phi_{2r+1}(t) = \sqrt{2}\cos(2\pi rt), r = 1, 2, \ldots, (m - 1)/2. \]

Eigenvalues \( \lambda_r = \nu \rho^r \), for some \( \nu > 0 \) and \( 0 < \rho < 1 \).

We set \( \xi_{ijkr} = \sqrt{\lambda_r} z_{ijkr} \) where \( z_{ijkr} \) are i.i.d random variables. To generate subject-effect functions \( v_{ijk}(t) \), we set

\[ z_{ijkr} \sim \text{i.i.d.} \ N(0, 1), \text{ or } z_{ijkr} \sim \text{i.i.d.} \ t_4/\sqrt{2}. \]  
(33)
Basis functions are

\[ \phi_1(t) = 1, \quad \phi_{2r}(t) = \sqrt{2}\sin(2\pi rt), \]
\[ \phi_{2r+1}(t) = \sqrt{2}\cos(2\pi rt), \quad r = 1, 2, \ldots, (m - 1)/2. \]

Eigenvalues \( \lambda_r = \nu \rho^r \), for some \( \nu > 0 \) and \( 0 < \rho < 1 \).

We set \( \xi_{i \tau i \tau k r} = \sqrt{\lambda_r} z_{i \tau i \tau k r} \) where \( z_{i \tau i \tau k r} \) are i.i.d random variables. To generate subject-effect functions \( v_{i \tau j k}(t) \), we set

\[ z_{i \tau i \tau k r} \sim \text{i.i.d. } N(0, 1), \text{ or } z_{i \tau i \tau k r} \sim \text{i.i.d. } t_4/\sqrt{2}. \quad (33) \]
Basis functions are

\[ \phi_1(t) = 1, \phi_{2r}(t) = \sqrt{2}\sin(2\pi rt), \]
\[ \phi_{2r+1}(t) = \sqrt{2}\cos(2\pi rt), r = 1, 2, \ldots, (m - 1)/2. \]

Eigenvalues \( \lambda_r = \nu \rho^r \), for some \( \nu > 0 \) and \( 0 < \rho < 1 \).

We set \( \xi_{ijkr} = \sqrt{\lambda_r} z_{ijkr} \) where \( z_{ijkr} \) are i.i.d random variables. To generate subject-effect functions \( v_{ijk}(t) \), we set

\[ z_{ijkr} \overset{i.i.d.}{\sim} N(0, 1), \text{ or } z_{ijkr} \overset{i.i.d.}{\sim} t_4/\sqrt{2}. \quad (33) \]
Result 1

Table: Empirical sizes and powers of $F$ and $T$ for testing $H_A$ when the functional data are Gaussian. The SAW was used.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta = 0$</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[10,5,10;5,10]</td>
<td>0.1</td>
<td>$F$</td>
<td>0.047</td>
<td>0.305</td>
<td>0.911</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T$</td>
<td>0.055</td>
<td>0.325</td>
<td>0.921</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>$F$</td>
<td>0.045</td>
<td>0.093</td>
<td>0.238</td>
<td>0.510</td>
<td>0.762</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T$</td>
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<td>0.099</td>
<td>0.251</td>
<td>0.527</td>
<td>0.777</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>$F$</td>
<td>0.041</td>
<td>0.048</td>
<td>0.081</td>
<td>0.165</td>
<td>0.271</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T$</td>
<td>0.044</td>
<td>0.052</td>
<td>0.086</td>
<td>0.176</td>
<td>0.284</td>
</tr>
<tr>
<td>[10,20,10;20,10]</td>
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<td>$F$</td>
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<td>0.598</td>
<td>0.999</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T$</td>
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<td>1.000</td>
</tr>
<tr>
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<td>0.984</td>
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<tr>
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<td></td>
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<tr>
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<td>0.355</td>
<td>0.606</td>
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<tr>
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<td>$F$</td>
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<td>0.164</td>
<td>0.549</td>
<td>0.911</td>
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<tr>
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<td>$T$</td>
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<td>$T$</td>
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<td>0.078</td>
<td>0.202</td>
<td>0.435</td>
<td>0.723</td>
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</table>
### Table: Empirical sizes and powers of $F$ and $T$ for testing $H_A$ when the functional data are Gaussian. The EQW was used.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta = 0$</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
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</tr>
<tr>
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<td></td>
<td>$T$</td>
<td>0.058</td>
<td>0.343</td>
<td>0.923</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>$F$</td>
<td>0.051</td>
<td>0.093</td>
<td>0.239</td>
<td>0.500</td>
<td>0.772</td>
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<td>$T$</td>
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<td>0.784</td>
</tr>
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<td>0.9</td>
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<td>0.044</td>
<td>0.086</td>
<td>0.169</td>
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</tr>
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<td>0.093</td>
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<td>0.083</td>
<td>0.202</td>
<td>0.426</td>
<td>0.713</td>
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</table>
**Table:** Empirical sizes and powers of $F$ and $T$ for testing $H_{\mathcal{A} \ast \mathcal{B}}$ when the functional data are Gaussian.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta = 0$</th>
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<th>0.6</th>
<th>0.9</th>
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<tbody>
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<td>[10,5,10;5,10,5]</td>
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<td>0.056</td>
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<td>0.064</td>
<td>0.071</td>
<td>0.099</td>
<td>0.147</td>
</tr>
<tr>
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<td>0.9</td>
<td>$F$</td>
<td>0.042</td>
<td>0.040</td>
<td>0.041</td>
<td>0.040</td>
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<td>0.047</td>
<td>0.046</td>
<td>0.048</td>
<td>0.064</td>
</tr>
<tr>
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<td>0.104</td>
<td>0.318</td>
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<td>0.333</td>
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<td>0.090</td>
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<tr>
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<td>0.9</td>
<td>$F$</td>
<td>0.046</td>
<td>0.047</td>
<td>0.056</td>
<td>0.067</td>
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<td>0.049</td>
<td>0.059</td>
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<td>[20,40,20;40,20,40]</td>
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<td>0.176</td>
<td>0.301</td>
</tr>
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<td>0.065</td>
<td>0.110</td>
<td>0.180</td>
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<tr>
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<td>0.9</td>
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<td>0.050</td>
<td>0.061</td>
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<td>0.052</td>
<td>0.064</td>
<td>0.080</td>
<td>0.116</td>
</tr>
</tbody>
</table>
In terms of power, the power of $T$ is higher than that of $F$.

In terms of size controlling, both tests are close to nominal size 5%. When $\rho = 0.1$ and $\rho = 0.5$, the $F$-type test outperformed the $L^2$-norm based test and when $\rho = 0.9$, the $L^2$-norm based test performed slightly better than the $F$-type test.

Two weighting system seems comparable. The SAW system seems more preferable in terms of size controlling especially $\rho$ is large.

As $\rho$ increases, the empirical sizes and powers of the two tests decrease. It is reasonable since when $\rho$ is large, the variations of the simulated functional data are large so that it is more challenging to maintain the nominal size and to detect the information.
In terms of power, the power of $T$ is higher than that of $F$.

In terms of size controlling, both tests are close to nominal size 5%. When $\rho = 0.1$ and $\rho = 0.5$, the $F$-type test outperformed the $L^2$-norm based test and when $\rho = 0.9$, the $L^2$-norm based test performed slightly better than the $F$-type test.

Two weighting system seems comparable. The SAW system seems more preferable in terms of size controlling especially $\rho$ is large.

As $\rho$ increases, the empirical sizes and powers of the two tests decrease. It is reasonable since when $\rho$ is large, the variations of the simulated functional data are large so that it is more challenging to maintain the nominal size and to detect the information.
In terms of power, the power of $T$ is higher than that of $F$.

In terms of size controlling, both tests are close to nominal size 5%. When $\rho = 0.1$ and $\rho = 0.5$, the $F$-type test outperformed the $L^2$-norm based test and when $\rho = 0.9$, the $L^2$-norm based test performed slightly better than the $F$-type test.

Two weighting system seems comparable. The SAW system seems more preferable in terms of size controlling especially $\rho$ is large.

As $\rho$ increases, the empirical sizes and powers of the two tests decrease. It is reasonable since when $\rho$ is large, the variations of the simulated functional data are large so that it is more challenging to maintain the nominal size and to detect the information.
In terms of power, the power of $T$ is higher than that of $F$.

In terms of size controlling, both tests are close to nominal size 5%. When $\rho = 0.1$ and $\rho = 0.5$, the $F$-type test outperformed the $L^2$-norm based test and when $\rho = 0.9$, the $L^2$-norm based test performed slightly better than the $F$-type test.

Two weighting system seems comparable. The SAW system seems more preferable in terms of size controlling especially $\rho$ is large.

As $\rho$ increases, the empirical sizes and powers of the two tests decrease. It is reasonable since when $\rho$ is large, the variations of the simulated functional data are large so that it is more challenging to maintain the nominal size and to detect the information.
Result 2

Table: Empirical sizes and powers of $F$ and $T$ for testing $H_A$ when the functional data are non-Gaussian. The SAW was used.

<table>
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<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta = 0$</th>
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<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[10,5,10;5,10]</td>
<td>0.1</td>
<td>$F$</td>
<td>0.051</td>
<td>0.099</td>
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<tr>
<td></td>
<td></td>
<td>$T$</td>
<td>0.057</td>
<td>0.108</td>
<td>0.306</td>
<td>0.613</td>
<td>0.911</td>
</tr>
<tr>
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<td>$F$</td>
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<td>0.098</td>
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<td>0.059</td>
<td>0.151</td>
<td>0.339</td>
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<tr>
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<td>0.951</td>
<td>0.999</td>
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<td>0.295</td>
<td>0.908</td>
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</table>

Liang Xuehua

On Some Hypothesis Testing Problems in Functional Data Analysis
**Table:** Empirical sizes and powers of $F$ and $T$ for testing $H_A$ when the functional data are non-Gaussian. The EQW was used.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta = 0$</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
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<tbody>
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<td>[10,5,10;5,10,5]</td>
<td>0.1</td>
<td>$F$</td>
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<td>$T$</td>
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<td>$T$</td>
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<td>0.105</td>
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<td>$T$</td>
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</table>
A GPF test for one-way ANOVA problem

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Two Sample Behrens-Fisher problem for functional data

Concluding Remarks and future work

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<table>
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<tr>
<th>Sample size</th>
<th>$\rho$</th>
<th>Method</th>
<th>$\delta$ = 0</th>
<th>0.3</th>
<th>0.6</th>
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<td>0.081</td>
<td>0.231</td>
<td>0.530</td>
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</table>
A real data: left cingulum data

- The response measurements are the levels of Radial Diffusibility from 39 kids from 9 to 19 years old over the arc length of the left cingulum from -60 to 60.
- There are two factors that may affect the RD levels of the kids, GHR and AGE where GHR stands for "Genetic High Risk".
- When GHR=1, it means that the kid is from a family with at least 1 first degree relatives with schizophrenia disease and when GHR=0, the kid is from a normal family. We transform age into two groups: kids of age 9-14 years old and kids of age 15-19 years old.
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$F$-type test under Gaussian assumption
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Table: Two-way ANOVA for the left cingulum data by the $F$-type and $L^2$-norm based tests. The SAW and EQW systems are considered.

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<td>EQW</td>
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<tr>
<td>GHR*AGE</td>
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Outline

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  Null distribution approximation and asymptotic power
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On Some Hypothesis Testing Problems in Functional Data Analysis
Two Sample Behrens-Fisher problem for functional data

The problem:

- Suppose that we have the following two independent functional samples:

\[ y_{l1}(t), \ldots, y_{ln_l}(t) \sim^i^d \text{GP}(\mu_l, \gamma_l), \quad l = 1, 2, \quad (34) \]

but we do not know if \( \gamma_1(s, t) \) and \( \gamma_2(s, t) \) are equal and we want to test

\[ H_0 : \mu_1(t) = \mu_2(t). \quad (35) \]
When the equality of the covariance functions can be assumed, the existing $L^2$-norm based test (Faraway 1997, Zhang and Chen 2007, Zhang, Peng and Zhang 2010), namely $ECL^2$, or the $F$-type test (Shen and Faraway 2004, Zhang 2011), namely $ECF$, can be adaptively applied.

When the equality is violated, or unknown, we aim to further investigate the $L^2$-norm based test, named as $UCL^2$ test.
When the equality of the covariance functions can be assumed, the existing $L^2$-norm based test (Faraway 1997, Zhang and Chen 2007, Zhang, Peng and Zhang 2010), namely ECL$^2$, or the $F$-type test (Shen and Faraway 2004, Zhang 2011), namely ECF, can be adaptively applied.

When the equality is violated, or unknown, we aim to further investigate the $L^2$-norm based test, named as UCL$^2$ test.
Set $z(t) = \sqrt{n} [\bar{y}_1(t) - \bar{y}_2(t)]$ where $\bar{y}_i = \sum_{j=1}^{n_i} y_{ij}(t) / n_i$ and $n = n_1 + n_2$.

The test statistic is given as

$$T_n = \int_{\mathcal{T}} z(t)^2 \, dt = n \int_{\mathcal{T}} \left[\bar{y}_1(t) - \bar{y}_2(t)\right]^2 \, dt.$$ (36)
Set $z(t) = \sqrt{n} [\bar{y}_1(t) - \bar{y}_2(t)]$ where $\bar{y}_i = \sum_{j=1}^{n_i} y_{ij}(t)/n_i$ and $n = n_1 + n_2$.

The test statistic is given as

$$T_n = \int_{\mathcal{I}} z(t)^2 dt = n \int_{\mathcal{I}} [\bar{y}_1(t) - \bar{y}_2(t)]^2 dt. \quad (36)$$
For any $t \in \mathcal{T}$, we have

$$z(t) - \mu_z(t) \xrightarrow{d} R(t),$$
$$\mu_z(t) = \sqrt{n} [\mu_1(t) - \mu_2(t)],$$

where $R(t) \sim \text{GP}(0, \gamma_z)$, $\tau = \lim_{n \to \infty} n_1/n$, and

$$\gamma_z(s, t) = \frac{\gamma_1(s, t)}{\tau} + \frac{\gamma_2(s, t)}{1-\tau}$$

$\gamma_z(s, t)$ has the following singular value decomposition:

$$\gamma_z(s, t) = \sum_{r=1}^{m} \lambda_r \phi_r(s) \phi_r(t).$$
For any $t \in T$, we have

$$z(t) - \mu_z(t) \overset{d}{\longrightarrow} R(t),$$

$$\mu_z(t) = \sqrt{n} [\mu_1(t) - \mu_2(t)],$$

where $R(t) \sim \text{GP}(0, \gamma_z)$, $\tau = \lim_{n \to \infty} n_1/n$, and

$$\gamma_z(s, t) = \frac{\gamma_1(s, t)}{\tau} + \frac{\gamma_2(s, t)}{1-\tau}.$$

$\gamma_z(s, t)$ has the following singular value decomposition:

$$\gamma_z(s, t) = \sum_{r=1}^{m} \lambda_r \phi_r(s) \phi_r(t).$$
Theorem

Assume that $\text{tr}(\gamma_l) < \infty$, $l = 1, 2$ and $0 < \tau < 1$. Then as $n \to \infty$, we have

$$T_n \xrightarrow{\text{asym}} \sum_{r=1}^{m} \lambda_r A_r + n \sum_{r=m+1}^{\infty} \pi_r^2,$$

where $A_r \sim \chi^2_{1}(n\lambda_r^{-1}\pi_r^2)$, $r = 1, \ldots, m$ are independent $\chi^2$ random variables, and $\pi_r = \int_{T} [\mu_1(t) - \mu_2(t)] \phi_r(t) dt$, $r = 1, \ldots, \infty$. 

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Firstly, under $H_0$, we have $\pi_r = 0, r = 1, \ldots, \infty$, which implies that the dominant term of $T_n$ under $H_0$ can be written as:

$$T^* \overset{d}{=} m \sum_{r=1}^{\infty} \lambda_r A_r, \quad A_r \overset{i.i.d.}{\sim} \chi_1^2.$$  \hfill (40)

Secondly, it is seen that when $\pi_r = 0, r = 1, \ldots, m$, we have $\sum_{r=m+1}^{\infty} \pi_r^2 = \sum_{r=1}^{\infty} \pi_r^2 = \|\mu_1 - \mu_2\|^2$. Therefore, we can simplify the random expression of $T_n$ as

$$T_n \overset{asym}{\sim} T^* + n\|\mu_1 - \mu_2\|^2.$$  \hfill (41)
Firstly, under $H_0$, we have $\pi_r = 0$, $r = 1, \ldots, \infty$, which implies that the dominant term of $T_n$ under $H_0$ can be written as:

$$T^* \overset{d}{=} \sum_{r=1}^{m} \lambda_r A_r, \quad A_r \overset{i.i.d.}{\sim} \chi_1^2. \quad (40)$$

Secondly, it is seen that when $\pi_r = 0$, $r = 1, \ldots, m$, we have $\sum_{r=m+1}^{\infty} \pi_r^2 = \sum_{r=1}^{m} \pi_r^2 = \|\mu_1 - \mu_2\|^2$. Therefore, we can simplify the random expression of $T_n$ as

$$T_n \overset{asym}{\sim} T^* + n\|\mu_1 - \mu_2\|^2. \quad (41)$$
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Under the null hypothesis, we can approximate the distribution of $T^*$ by that of a random variable of the form $R = \beta \chi^2_d$. The parameters $\beta$ and $d$ are determined as

$$\beta = \frac{\sum_{r=1}^{m} \lambda_r^2}{\sum_{r=1}^{m} \lambda_r} = \frac{\text{tr}(\gamma \otimes^2)}{\text{tr}(\gamma)}$$

$$d = \frac{(\sum_{r=1}^{m} \lambda_r)^2}{\sum_{r=1}^{m} \lambda_r^2} = \frac{\text{tr}^2(\gamma_z)}{\text{tr}(\gamma \otimes^2)}.$$
The asymptotic unbiased estimator of $\hat{\gamma}_z(s, t)$ is

$$
\hat{\gamma}_z(s, t) = n \left[ \hat{\gamma}_1(s, t)/n_1 + \hat{\gamma}_2(s, t)/n_2 \right],
$$

(42)

where $\hat{\gamma}_l(s, t) = \sum_{j=1}^{n_l} [y_{lj}(s) - \bar{y}_l(s)] [y_{lj}(t) - \bar{y}_l(t)] / (n_l - 1)$, $l = 1, 2$, are the unbiased estimators of $\gamma_l(s, t)$, $l = 1, 2$ respectively.

Further we have $\hat{\beta} = \frac{\text{tr}(\hat{\gamma}_z \otimes^2)}{\text{tr}(\hat{\gamma}_z)}$, $\hat{d} = \frac{\text{tr}^2(\hat{\gamma}_z)}{\text{tr}(\hat{\gamma}_z \otimes^2)}$. It follows that the $\alpha$-level critical value of $T_n$ can be approximately specified as $\hat{T}_\alpha = \hat{\beta} \chi^2_{d, \alpha}$, where $\chi^2_{d, \alpha}$ denotes the upper 100$\alpha$ percentile of $\chi^2_d$. 
The asymptotic unbiased estimator of $\hat{\gamma}_z(s, t)$ is

$$
\hat{\gamma}_z(s, t) = n \left[ \hat{\gamma}_1(s, t)/n_1 + \hat{\gamma}_2(s, t)/n_2 \right],
$$

(42)

where $\hat{\gamma}_l(s, t) = \sum_{j=1}^{n_l} \left[ y_{lj}(s) - \bar{y}_l(s) \right] \left[ y_{lj}(t) - \bar{y}_l(t) \right] / (n_l - 1), l = 1, 2$, are the unbiased estimators of $\gamma_l(s, t), l = 1, 2$ respectively.

Further we have $\hat{\beta} = \frac{\text{tr}(\hat{\gamma}_z \otimes 2)}{\text{tr}(\hat{\gamma}_z)}, \hat{d} = \frac{\text{tr}^2(\hat{\gamma}_z)}{\text{tr}(\hat{\gamma}_z \otimes 2)}$. It follows that the $\alpha$-level critical value of $T_n$ can be approximately specified as $\hat{T}_\alpha = \hat{\beta} \chi^2_{d, \alpha},$ where $\chi^2_{d, \alpha}$ denotes the upper 100$\alpha$ percentile of $\chi^2_d$. 
Theorem

Assume that $\mathcal{T}$ is a finite interval, $tr(\gamma_l) < \infty$, $l = 1, 2$ and $0 < \tau < 1$. Then as $n \to \infty$, we have $\hat{\beta} \xrightarrow{a.s.} \beta$, $\hat{d} \xrightarrow{a.s.} d$, and $\hat{T}_\tau \xrightarrow{a.s.} T^0_\tau$. 
\textbf{Theorem}

Assume that $\mathcal{T}$ is a finite interval, $\text{tr}(\gamma_l) < \infty$, $l = 1, 2$ and $0 < \tau < 1$. Then as $n \to \infty$, we have $\hat{\beta} \xrightarrow{\text{a.s.}} \beta$, $\hat{d} \xrightarrow{\text{a.s.}} d$, and $\hat{T}_\alpha \xrightarrow{\text{a.s.}} T^0_\alpha$. 
Asymptotic Power under Local Alternatives

- When the alternative is fixed, it is easy to show that the associated power must tend to 1 as $n \to \infty$.

- We will study the power behavior of $T_n$ when the alternatives are tending to the null hypothesis with a rate slightly slower than $n^{-1/2}$. We define a sequence of local alternatives as:

$$H_{1n} : \mu_1(t) - \mu_2(t) = n^{-\omega/2} u(t), \quad (43)$$

where $\omega$ is some constant satisfying $0 \leq \omega < 1$ and $u(t)$ is any fixed real function such that $0 < \int_T u^2 dt < \infty$. 
Asymptotic Power under Local Alternatives

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Theorem

If \( n_1 \) and \( n_2 \) tend to \( \infty \) proportionally, the asymptotic power of \( T_n \)

\[
P \left( T_n \geq \hat{T}_\alpha | H_{1n} \right) \to 1.
\]
A simulation

- Simulations are conducted to compare the empirical sizes (Type-I error rates) and powers of our testing procedure \( \text{UCL}^2 \) with those of the two existing testing procedures: ECF and ECL^2.

- Two functional samples of sizes \( n_1 \) and \( n_2 \) are generated with
  \[ \mu_l(t) = \mathbf{c}_l^T [1, t, t^2, t^3]^T, \ l = 1, 2 \]
  \[ \gamma_l(s, t) = \sum_{r=1}^{q_l} \lambda_{lr} \psi_{lr}(s) \psi_{lr}(t), \ l = 1, 2 \]
  where \( \mathbf{c}_l = [c_{l0}, c_{l1}, c_{l2}, c_{l3}]^T \) are constant coefficient vectors, \( \psi_{l1}(t), \psi_{l2}(t), \ldots, \psi_{lq_l}(t), t \in [0, 1], l = 1, 2 \) are the bases and \( \lambda_{l1}, \lambda_{l2}, \ldots, \lambda_{lq_l}, l = 1, 2 \) are nonnegative variance components.
A simulation

Simulations are conducted to compare the empirical sizes (Type-I error rates) and powers of our testing procedure UCL$^2$ with those of the two existing testing procedures: ECF and ECL$^2$.

Two functional samples of sizes $n_1$ and $n_2$ are generated with $\mu_l(t) = c_l^T [1, t, t^2, t^3]^T, l = 1, 2$ and $\gamma_l(s, t) = \sum_{r=1}^{q_l} \lambda_{lr} \psi_{lr}(s) \psi_{lr}(t), l = 1, 2$ where $c_l = [c_{l0}, c_{l1}, c_{l2}, c_{l3}]^T$ are constant coefficient vectors, $\psi_{l1}(t), \psi_{l2}(t), \cdots, \psi_{lq_l}(t), t \in [0, 1], l = 1, 2$ are the bases and $\lambda_{l1}, \lambda_{l2}, \cdots, \lambda_{lq_l}, l = 1, 2$ are nonnegative variance components.
For \( l = 1, 2 \), we set \( \lambda_{lr} = a_l \rho_l^r \), \( r = 1, \ldots, q_l \) for some \( a_l > 0 \) and \( 0 < \rho_l < 1 \). Without loss of generality, we set \( a_1 = 1 \), \( \rho_1 = \rho_2 = .7 \), and \( q_1 = q_2 = 11 \) and both the bases are the cosine-sine basis.

Without loss of generality, we randomly generate \( c_1 \) and specify \( c_2 = c_1 + \delta 1_4 \).

The remaining tuning parameters such as \( a_2 \). When \( a_2 = 1 \), the two samples have the same covariance functions and when \( a_2 = 5, 10 \), the two samples have different covariance functions.
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Null distribution approximation and asymptotic power
Simulation and Real data analysis

**Table:** Empirical sizes and powers of the three testing procedures when the sample sizes are equal.

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### Table: Empirical sizes and powers of the three testing procedures when the sample sizes are moderate and unequal.

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The problem and test statistics
Null distribution approximation and asymptotic power
Simulation and Real data analysis

Table: Empirical sizes and powers of the three testing procedures when the sample sizes are large and unequal.

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<td>.0054</td>
<td>.0102</td>
<td>.0299</td>
<td>.0778</td>
<td>.1653</td>
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<tr>
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<td></td>
<td></td>
<td>UCL^2</td>
<td>.0501</td>
<td>.0606</td>
<td>.1264</td>
<td>.2464</td>
<td>.4023</td>
<td>.5786</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>30</td>
<td>ECF</td>
<td>.1845</td>
<td>.2031</td>
<td>.2728</td>
<td>.3742</td>
<td>.5152</td>
<td>.6448</td>
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<tr>
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<td>ECL^2</td>
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<td>.0567</td>
<td>.0984</td>
<td>.1620</td>
<td>.2722</td>
<td>.3970</td>
</tr>
</tbody>
</table>
Berkeley Growth Data

- The heights of 39 boys and 54 girls were recorded at 31 not equally spaced ages from Year 1 to Year 18.
- We are interested in testing if the boys and the girls have the same mean heights over the period from Year 1 to Year 18 as well as in some specified growth periods between Year 1 and Year 18.
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Table: $P$-values for significance tests of the mean growth curves of boys and girls of the Berkeley growth data (the resolution dimension $M = 1000$).

<table>
<thead>
<tr>
<th>Period $[a, b]$</th>
<th>$n_1 = 39, n_2 = 54$</th>
<th>$n_1 = 10, n_2 = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ECF</td>
<td>ECL$^2$</td>
</tr>
<tr>
<td>$[1, 4)$</td>
<td>$6.72 \times 10^{-3}$</td>
<td>$5.64 \times 10^{-3}$</td>
</tr>
<tr>
<td>$[4, 13]$</td>
<td>$3.16 \times 10^{-1}$</td>
<td>$3.14 \times 10^{-1}$</td>
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<tr>
<td>$[13, 18]$</td>
<td>$7.21 \times 10^{-11}$</td>
<td>$2.64 \times 10^{-13}$</td>
</tr>
<tr>
<td>$[1, 18]$</td>
<td>$1.05 \times 10^{-6}$</td>
<td>$2.23 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Concluding Remarks

- A GPF test was firstly proposed for one-way ANOVA models via globalizing the classical pointwise $F$-test in section 2. As a nice summary of the pointwise $F$-type test of Ramsay and Silverman (2005), the GPF test is useful to give a global assessment for the main effect of the factor in the one-way ANOVA problem. The GPF test is also applied to the Canada climate data (1982).
Then the $L^2$-norm test and $F$-type test were proposed and studied for two-way ANOVA models for Gaussian and non-Gaussian functional data in Section 3. We studied how to form the test statistics, how to approximate the null distributions of the two tests and studied their asymptotical powers. We also conducted intensive simulation studies to compare the two tests under Gaussian and non-Gaussian assumptions. The methodologies are applied to left cingulum data.
We proposed the unequal covariance $L^2$-norm based test, namely, UCL$^2$ test for two sample Behrens-Fisher problem for functional data in Section 4. By some simulations, we see that when the covariance function homogeneity assumption is not satisfied, the UCL$^2$ test outperformed the usual $L^2$-norm based and $F$-type tests. The UCL$^2$ test is also used to Berkeley growth data.
Future work:

- For functional ANOVA problem, the GPF test, \( F \)-type test and \( L^2 \)-norm test can be extended to more complex models such as higher-way ANOVA.

- For functional Behrens-Fisher problem, it is more interesting and surely more challenging to study the case of \( k \)-sample problem with \( k > 2 \).

- Another future direction is to study the functional linear model with continuous covariates such that

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y_i(t) = x_i^T(t)\beta(t) + v_i(t), \quad v_i(t) \sim \text{RP}(0, \gamma).
\]
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Reference

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Thank you!