TUTORIAL 4 SOLUTIONS

#8.10.12 Suppose that you had to choose either the method of moments estimates or the maximum likelihood estimates in Example C of Section 8.4 and C of Section 8.5. Which would you choose and why?

Solution We shall focus on Example C of Section 8.4 as the same reasons apply to Example C of Section 8.5.

The method of moments estimates for the parameters $\lambda$ and $\alpha$ are available in closed-form:

$$\hat{\lambda} = \frac{\bar{X}}{\hat{\sigma}^2},$$

$$\hat{\alpha} = \frac{\bar{X}^2}{\hat{\sigma}^2}.$$ 

This is typical for method of moments estimates and is an attractive property of method of moments estimates.
On the other hand, the mle’s for \( \lambda \) and \( \alpha \) are not available in closed-form.

However it can be shown by simulations (and also theoretically) that the sampling distributions of the two mle’s are substantially less dispersed than those of the method of moments estimates.

This means that the two mle’s are generally more accurate than the method of moments estimates for \( \lambda \) and \( \alpha \).
Suppose that $X_1, \ldots, X_n$ are i.i.d. with density function

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta,$$

and $f(x|\theta) = 0$ otherwise.

a. Find the method of moments estimate of $\theta$.

b. Find the mle of $\theta$. (*Hint:* Be careful, don’t differentiate before thinking. For what values of $\theta$ is the likelihood positive?)

c. Find a sufficient statistic for $\theta$.

**Solution**

a. Let $\mu = E(X_1)$. Then

$$\mu = \int_{\theta}^{\infty} x e^{-(x-\theta)} \, dx$$

$$= \int_{0}^{\infty} (x + \theta) e^{-x} \, dx$$

$$= \theta + 1.$$
Thus $\theta = \mu - 1$ and the method of moments estimate of $\theta$ is

$$\hat{\theta} = \hat{\mu} - 1 = \bar{X} - 1.$$ 

b. Note that the density function $f(x|\theta)$ is not differentiable at $x = \theta$.

Also $f(x|\theta)$ is strictly decreasing for $x \geq \theta$ and equals 0 for $x < \theta$.

The likelihood function is

$$\text{lik}(\theta) = \prod_{i=1}^{n} e^{-(x_i - \theta)}$$

if $\min(x_1, \ldots, x_n) \geq \theta$, and $\text{lik}(\theta) = 0$ otherwise.

Hence the maximum of $\text{lik}(\theta)$ occurs at $\theta = \min(x_1, \ldots, x_n)$ and we conclude that the mle of $\theta$ is

$$\tilde{\theta} = \min(X_1, \ldots, X_n).$$
c. Let $I\{x_i \geq \theta\}$ denote the indicator function of the event \{\(x_i \geq \theta\)\}. 

I.e., $I\{x_i \geq \theta\} = 1$ if $x_i \geq \theta$ and $I\{x_i \geq \theta\} = 0$ if $x_i < \theta$.

Then the joint pdf of $X$ is given by

$$f(x|\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I\{x_i \geq \theta\}$$

$$= e^{-\sum_{i=1}^{n} x_i} e^{n\theta} I\{\min(x_1, \ldots, x_n) \geq \theta\}$$

$$= g(t, \theta)h(x),$$

where

$$t = \min(x_1, \ldots, x_n),$$

$$g(t, \theta) = e^{n\theta} I\{\min(x_1, \ldots, x_n) \geq \theta\},$$

$$h(x) = e^{-\sum_{i=1}^{n} x_i}.$$

It follows from the factorization theorem that $T = \min(X_1, \ldots, X_n)$ is sufficient for $\theta$. 
George spins a coin three times and observes no heads. He then gives the coin to Hilary. She spins it until the first head occurs, and ends up spinning it four times total. Let $\theta$ denote the probability that the coin comes up heads.

a. What is the likelihood of $\theta$?

b. What is the MLE of $\theta$?

Solution

a. The likelihood function is

$$\text{lik}(\theta) = P(\text{heads})[P(\text{tails})]^6$$

$$= \theta(1 - \theta)^6.$$
b. The loglikelihood is

\[ l(\theta) = \log(\theta) + 6 \log(1 - \theta). \]

Differentiating \( l(\theta) \) wrt \( \theta \) and then solving for \( l'(\theta) = 0 \), we get

\[ l'(\theta) = \frac{1}{\theta} - \frac{6}{1 - \theta} = 0. \]

The solution for (1) is \( \theta = 1/7 \). Hence we conclude that the MLE of \( \theta \) is

\[ \hat{\theta} = 1/7. \]