#6.4.1 Prove Proposition A of Section 6.2.

**Solution** Recall that the random variable $T$ is said to have the $t_n$ distribution if

$$T = \frac{Z}{\sqrt{U/n}}$$

where $Z \sim N(0, 1)$, $U \sim \chi^2_n$ and $Z, U$ are independent.

Fix $U = u$. Then by independence of $Z, U$, the conditional pdf of $Z$ given $U$ is

$$f_{Z|U}(z|u) = f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let $T = h(Z) = Z/c$ where $c = \sqrt{u/n}$. Then $Z = h^{-1}(T) = Tc$ which implies that $dZ/dT = c$. 

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Consequently using the change of variables formula, we obtain

\[ f_{T|U}(t|u) = f_{Z|U}(h^{-1}(t)|u) \left| \frac{dZ}{dT} \right|. \]

Since

\[ f_{Z|U}(h^{-1}(t)|u) = f_Z(h^{-1}(t)) = \frac{1}{\sqrt{2\pi}} e^{-t^2c^2/2}, \]

we have

\[ f_{T|U}(t|u) = \frac{c}{\sqrt{2\pi}} e^{-t^2c^2/2} = \frac{\sqrt{u}}{\sqrt{2\pi n}} e^{-t^2u/(2n)}. \]

Also we observe that the pdf of \( U \) is

\[ f_U(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2}, \quad u \geq 0. \]
This implies that the joint pdf of \((T, U)\) is

\[
\begin{align*}
    f_{T,U}(t, u) &= f_{T|U}(t|u)f_U(u) \\
    &= \frac{\sqrt{u}}{\sqrt{2\pi n}} e^{-t^2u/(2n)} \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \\
    &= \frac{2^{-(n+1)/2} u^{(n-1)/2} \exp\left\{-\frac{u}{2(1 + t^2/n)^{-1}}\right\}}{\sqrt{\pi n \Gamma(n/2)}},
\end{align*}
\]
for all \(u > 0\) and \(t \in \mathbb{R}\).
Hence

\[ f_T(t) = \int_0^\infty f_{T,U}(t, u) \, du \]

\[ = 2^{-(n+1)/2} \frac{\sqrt{\pi n \Gamma(n/2)}}{\sqrt{\pi n \Gamma(n/2)}} \times \int_0^\infty u^{(n-1)/2} \exp\left\{-\frac{u}{2(1 + t^2/n)}\right\} du. \]

Writing \( 2/(1 + t^2/n) = a \) and \( v = u/a \), we have

\[ f_T(t) = 2^{-(n+1)/2} \frac{\sqrt{\pi n \Gamma(n/2)}}{\sqrt{\pi n \Gamma(n/2)}} a^{(n+1)/2} \int_0^\infty (av)^{(n-1)/2} e^{-v} \, adv \]

\[ = 2^{-(n+1)/2} \frac{\sqrt{\pi n \Gamma(n/2)}}{\sqrt{\pi n \Gamma(n/2)}} a^{(n+1)/2} \int_0^\infty v^{(n+1)/2 - 1} e^{-v} \, dv \]

\[ = 2^{-(n+1)/2} a^{(n+1)/2} \Gamma\left(\frac{n + 1}{2}\right), \]
since
\[ \Gamma\left(\frac{n + 1}{2}\right) = \int_0^\infty v^{(n+1)/2-1} e^{-v} dv. \]

Thus we conclude that
\[ f_T(t) = \frac{\Gamma[(n + 1)/2]}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}. \]
#6.4.2 Prove Proposition B of Section 6.2.

**Solution** Recall that $W$ is said to have a $F_{m,n}$ distribution if

$$W = \frac{U/m}{V/n}$$

where $U \sim \chi^2_m$, $V \sim \chi^2_n$ and $U, V$ are independent.

First fix $V = v$. Then

$$f_{U|V}(u|v) = f_U(u) = \frac{1}{2^{m/2}\Gamma(m/2)}u^{(m/2)-1}e^{-u/2}.$$  

We write

$$W = h(U) = \frac{U/m}{v/n} = cU,$$

where $c = n/(mv)$. Then

$$U = h^{-1}(W) = W/c.$$
This implies that
\[
    f_{W|V}(w|v) = f_{U|V}(h^{-1}(w)|v) \left| \frac{dU}{dW} \right|
\]
\[
= \frac{1}{c} f_{U|V}(\frac{w}{c}|v)
\]
\[
= \frac{1}{c} f_{U}(\frac{w}{c})
\]
\[
= \frac{1}{2^{m/2} \Gamma(m/2) c} \left( \frac{w}{c} \right)^{(m/2)-1} e^{-w/(2c)}
\]
\[
= \frac{1}{\Gamma(m/2) \frac{m}{2n}} \left( \frac{m}{2n} \right)^{m/2} w^{m/2} e^{-\frac{mv}{2} - wvm/(2n)}.
\]

Since
\[
f_{V}(v) = \frac{1}{2^{n/2} \Gamma(n/2)} v^{(n/2)-1} e^{-v/2},
\]
we have
\[ \begin{align*}
  f_{W,V}(w, v) & = f_{W|V}(w|v) f_V(v) \\
  & = \frac{1}{\Gamma(m/2) 2n} (\frac{m}{n})^{m/2} w^{(m/2)-1} v^{m/2} e^{-wvm/(2n)} \\
  & \quad \times \frac{1}{2^{n/2} \Gamma(n/2)} v^{(n/2)-1} e^{-v/2} \\
  & = \frac{1}{\Gamma(m/2) \Gamma(n/2) 2^{(m+n)/2}} \\
  & \quad \times (\frac{m}{n})^{m/2} w^{(m/2)-1} v^{(m+n)/2-1} \\
  & \quad \times \exp\{-v(\frac{wm}{2n} + \frac{1}{2})\}, \quad w, v > 0. \\
\end{align*} \]

Finally writing \( a = 1/(\frac{wm}{2n} + \frac{1}{2}) \), we have
\[ f_W(w) = \int_0^\infty f_{W,V}(w,v)dv \]
\[ = \frac{1}{\Gamma(m/2) \Gamma(n/2) 2^{(m+n)/2} \left(\frac{m}{n}\right)^{m/2}} \times w^{(m/2)-1} \int_0^\infty v^{(m+n)/2-1} e^{-v/a} dv \]
\[ = \frac{1}{\Gamma(m/2) \Gamma(n/2) 2^{(m+n)/2} \left(\frac{m}{n}\right)^{m/2}} \times w^{(m/2)-1} \int_0^\infty (ax)^{(m+n)/2-1} e^{-x} dx \]
\[ = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2) \Gamma(n/2) 2^{(m+n)/2} \left(\frac{m}{n}\right)^{m/2}} \times w^{(m/2)-1} a^{(m+n)/2}, \]

since
\[ \Gamma[(m+n)/2] = \int_0^\infty x^{(m+n)/2-1} e^{-x} dx. \]
Consequently we conclude that

\[ f_W(w) = \frac{\Gamma((m + n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left( \frac{m}{n} \right)^{m/2} \]
\[ \times w^{(m/2)-1} (1 + \frac{m}{n}w)^{-(m+n)/2}. \]
#6.4.5 Show that if $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$.

Solution This is easy since

$$X = (U/m)/(V/n)$$

if and only if

$$X^{-1} = (V/n)/(U/m).$$

This implies from the definition of the $F$-distribution that $X \sim F_{m,n}$ if and only if $X^{-1} \sim F_{n,m}$. 
#6.4.6 Show that if \( T \sim t_n \), then \( T^2 \sim F_{1,n} \).

**Solution** If \( T \sim t_n \), then \( T \) can be expressed as

\[
T = \frac{Z}{\sqrt{V/n}}
\]

where \( Z \sim N(0, 1) \), \( V \sim \chi^2_n \) and \( Z, V \) are independent.

Consequently,

\[
T^2 = \frac{Z^2}{V/n} = \frac{U/1}{V/n}
\]

where \( U \sim \chi^2_1 \). This proves that \( T^2 \sim F_{1,n} \).
Show that the Cauchy distribution and the $t$ distribution with 1 degree of freedom are the same.

**Solution** Recall that the pdf of a Cauchy distribution is

$$f(x) = \frac{1}{\pi\left(1 + x^2\right)}, \quad x \in \mathbb{R}. $$

On the other hand, we observe from Theorem A of Chapter 6.2.1 that the pdf of $t_1$ is

$$g(x) = \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)}\left(\frac{1}{1 + x^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{1 + x^2}\right), \quad x \in \mathbb{R},$$

since $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

Thus the Cauchy distribution and the $t$ distribution with 1 degree of freedom are the same.