SINGULAR STOCHASTIC CONTROL
AND A MODIFIED BLACK-SCHOLES THEORY
INCORPORATING TRANSACTION COSTS*

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Abstract

In the presence of transaction costs, it is no longer possible to perfectly replicate the payoff of a European option by trading in the underlying stock. This paper develops a new option hedging strategy based on minimizing the expected cumulative hedging error and additional cost of rebalancing due to proportional transaction costs. We show that the resulting singular stochastic control problem is equivalent to an optimal stopping problem with relatively low computational complexity. Our results show that an optimal hedge consists of selling or buying the underlying stock whenever the holding of shares falls above or below a no-transaction band containing the option’s delta. Using a self-financing argument, we derive a writer’s and a buyer’s values of an option as the expected total cost of portfolio adjustments when the optimal hedge is carried out and then establish bounds on these values. A simulation study shows that our dynamic hedging strategy is more effective in reducing both the risk and cost of trading in options than discrete-time replicating strategies with prespecified revision times.

Key Words: Option hedging and pricing, singular stochastic control, optimal stopping, transaction costs

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1. INTRODUCTION

In the Black-Scholes option pricing model, where there are no transaction costs in buying or selling the underlying stock, the value of an option is equal to the value of a self-financing replicating portfolio comprising the stock and a risk-free bond; see Black and Scholes (1973). The Black-Scholes replicating strategy constitutes \textit{continuously} adjusting the portfolio to consist of delta shares of stock at any time, delta being the first partial derivative of the option price with respect to the stock price, thereby rendering the replicating portfolio completely riskless. This strategy is, however, difficult to follow in practice and frequently we find that options are hedged by adjusting the “replicating” portfolios at discrete time-points. Taking this discrete-time approach, the strategy involves purchasing delta shares of stock at the \textit{beginning of each time period}, thus replicating the call option’s payoff at expiration. Boyle and Emanuel (1980) studied the distribution of the difference between changes in values of the discrete-time replicating portfolio and changes in values of the call option (as given by the Black-Scholes formula). They found that its variance is roughly halved when the revision interval is halved, in agreement with simulation results of Leland (1985) for the mean and variance of this distribution; see in particular Table I of Leland (1985).

In the presence of transaction costs, Leland (1985) demonstrated that this discrete-time strategy is inadmissible, in the sense that transaction costs become arbitrarily large and the accuracy of replicating the option does not increase as the revision interval is shortened to zero; see in particular Table VII of Leland (1985). He proposed a modified strategy in which the delta is computed from the Black-Scholes option pricing formula with a modified variance so that for sufficiently short revision intervals, the discrete-time replicating portfolio yields the desired option payoff at expiration inclusive of transaction costs. This modified strategy, however, is not self-financing and also requires an exogenously specified revision interval by the user to implement the strategy. Moreover, the option becomes more expensive with more frequent revisions, so that in the limit as the time between trades approaches zero, the adjusted volatility approaches infinity and the option costs as much as a single share of stock. In fact, Soner et al. (1995) proved that if one is attempting to dominate the option rather than replicate it, the trivial strategy of buying one share of stock and holding it to maturity is the least expensive in a Black-Scholes model with proportional transaction costs.

Instead of using geometric Brownian motion as a model for stock prices, Boyle and Vorst (1992) assumed a discrete-state (binomial tree) framework to construct \textit{self-financing} discrete-time replicating portfolios, thereby extending the two-period model of Merton (1990, Chapter 14). They developed a backward induction procedure to compute explicit portfolio weights at each node of the binomial tree and showed that
for small proportional transaction costs and a large number of portfolio revisions, the option cost is again
the Black-Scholes value with a modified variance, but with a factor larger than that of Leland (1985). Be-
cause certain parameter combinations violate the necessary conditions for a unique solution, the recursive
procedure of Boyle and Vorst (1992) cannot be used to compute short call prices for these parameter values;
see in particular Table IV of Boyle and Vorst (1992). For the binomial tree model, however, Bensaid et al.
(1992) showed that it is possible to have a less expensive portfolio of stock and bond that dominates (“super-
replicates”) the option at maturity by rebalancing only in the initial periods. This is achieved by formulating
a cost minimization problem and using dynamic programming to solve it. Edirisinghe et al. (1993) noted
that this optimization problem is path-dependent and is computationally expensive if the number of periods
is not sufficiently small, leading them to develop a linear programming algorithm and a two-stage dynamic
programming method to approximate the optimal solution.

To address the problem of optimal hedging of an option by trading in the underlying stock, the replication
problem can be defined in terms of a utility function. In Hodges and Neuberger (1989), maximizing expected
exponential utility leads to a stochastic control problem which can be solved by dynamic programming.
Using this approach, it turns out that one should rebalance the portfolio only when the number of shares
of stock falls “too far” out of line (relative to a “target”). The procedure of Hodges and Neuberger (1989)
is, however, computationally expensive and liable to numerical instability. This has prompted Clewlow and
Hodges (1997) to develop an improved numerical scheme after a logarithmic transformation. Also taking
the utility maximization approach, Davis et al. (1993) introduced a modified definition of the price for the
option writer; this definition involves two stochastic control problems so that the price is characterized by
the indifference of the writer between going into the market to hedge the option and going into the market
without a position in the option.

In this paper, we develop a new approach to option hedging in the presence of transaction costs. Our
approach is based on minimizing the expected cumulative hedging error (a measure of “risk”) and additional
cost of rebalancing due to proportional transaction costs, taking the Black-Scholes value to be the market
price. Unlike the utility maximization approach, our formulation reduces to the option hedging problem of
Black and Scholes (1973) when transaction costs approach zero. An important feature of our formulation
of option hedging as a singular stochastic control problem is its connection to an “equivalent” optimal
stopping problem that is relatively easy to solve. Specifically, we make use of results from Karatzas and
Shreve (1985) on the reflected follower problem to establish that the no-action region of the original singular
stochastic control problem is given by the continuation region of an equivalent optimal stopping problem.
After deriving the optimal hedging strategy, we use a self-financing argument to derive upper (resp. lower)
bounds on the option value as the maximum (resp. minimum) expected initial capital required to finance
downstream rebalancing trades in the option writer’s (resp. buyer’s) optimal hedging strategy. We show that
our method of option hedging compares favorably with the discrete-time approach of Leland (1985) and
with the utility-based approach of Hodges and Neuberger (1989) and Clewlow and Hodges (1997) in the
mean-variance sense; see also Toft (1996).

The paper is organized as follows. We describe in Section 2 the singular stochastic control problem
associated with optimal hedging and show how it can be reformulated into an optimal stopping problem
whose solution is relatively easy to obtain by applying backward induction to an approximating Bernoulli
walk. We then derive in Section 3 the writer’s and buyer’s option values after incorporating the costs of
the optimal hedging strategies given in Section 2. We also show how these hedging costs can be computed
by using Monte Carlo simulation. In Section 4, we present several properties of the optimal hedge, such
as transaction frequencies and average trade sizes, and make comparison with the discrete-time hedging
strategy of Leland (1985) and the utility-based approach of Clewlow and Hodges (1997). Some extensions
of our method are described in Section 5 before we conclude in Section 6. Throughout the paper, the top
(resp. bottom) sign of ± and ∓ applies to the call (resp. put) option.

2. OPTION HEDGING WITH TRANSACTION COSTS

Consider an option written on a stock, with strike price $K$ and expiration date $T$. A call (resp. put) option
gives its holder the right to purchase (resp. sell) one share of stock for the price of $K$ at time $T$. We assume
that the option is settled by physical delivery of the stock and that the stock price process follows a geometric
Brownian motion

\[ dS_t = (\alpha - q)S_t \, dt + \sigma S_t \, dW_t, \]

where $\alpha$ and $\sigma$ are the mean and standard deviation (or volatility) of the stock’s return, $q$ is the dividend
rate paid by the underlying stock, and $\{W_t, \ t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$; cash
settlement is discussed in Section 5.2. Let $r$ be the riskless rate of return. It will be convenient to divide
all prices through by the strike price $K$ so the option strike price becomes 1. The option is to be hedged by
holding $y_t$ shares of the stock at time $t$, such that

\[ dy_t = dL_t - dM_t, \]

where $L_t$ (resp. $M_t$) is the cumulative number of shares of stock bought (resp. sold) up to time $t$. In the
presence of transaction costs proportional to the value of stock traded, the purchase of $dL_t$ shares of stock
would cost \((1 + \lambda)(S_t/K)\) \(dL_t\), while the sale of \(dM_t\) shares of stock would yield \((1 - \mu)(S_t/K)\) \(dM_t\), where \(0 \leq \lambda, \mu < 1\) are the proportional cost rates for the purchase and sale of stock, respectively. Thus, an additional “rebalancing” cost of \(\lambda(S_t/K)\) \(dL_t\) or \(\mu(S_t/K)\) \(dM_t\) is incurred on the purchase or sale of stock. The inclusion of fees proportional to the number of shares traded is straightforward and will be deferred to Section 5.1 to keep the notation here simple.

Consider first a short position in the option. The option writer’s hedging portfolio consists of \(-1\) option and \(y_t\) shares of stock at time \(t\). If the “market” price of the option is \(P(t, S_t)\), the value of the hedging portfolio is \(\Pi(t, S_t, y_t) = -P(t, S_t) + y_tS_t\). By (2.1) and Ito’s formula, \(\Pi\) satisfies

\[
d\Pi(t, S_t, y_t) = \mu\Pi(t, S_t, y_t)\,dt + \sigma S_t[y_t - \Delta(t, S_t)]\,dW_t,
\]

where \(\mu\Pi(t, S_t, y_t)\) is the mean return of the portfolio and \(\Delta(t, S_t) = \partial P(t, S_t) / \partial S\) is the “market” delta. Since \(P(T, S) = \{\pm(S - K)\}_+\), we must have \(\Delta(T, S) = \pm\mathbb{I}_{\{S > \pm K\}}\). After dividing all prices by \(K\), the instantaneous conditional variance of the hedging portfolio is \(F(t, S_t, y_t)\) \(dt\), where

\[
F(t, S_t, y_t) = \sigma^2(S_t/K)^2[y_t - \Delta(t, S_t)]^2.
\]

Because of transaction costs, continuously rebalancing to \(y_t = \Delta(t, S_t)\) shares of stock (thereby rendering the hedging portfolio completely riskless) is ruinously expensive and therefore not optimal. Instead, one should determine the stock holding \(y_t\) at time \(t\) that makes the hedging portfolio (of option and stock) as riskless as possible in the presence of transaction costs. This leads to the stochastic control problem of finding the hedging strategy that minimizes the expected cumulative conditional variance plus additional rebalancing cost due to transaction costs, given by

\[
J(t, S_t, y_t) = \mathbb{E}\left\{ \int_t^T F(u, S_u, y_u) \,du + \lambda \int_t^T (S_u/K) \,dL_u + \mu \int_t^T (S_u/K) \,dM_u \right\} \bigg| S_t = S, y_t = y.
\]

The optimal hedging strategy then corresponds to a pair of nondecreasing and non-anticipating processes \((L, M)\) that minimizes \(J(t, S_t, y_t)\). When an option is exercised against the writer, the writer is required to deliver \(y_T = \pm 1\) share of stock and receives a payment of \(\pm K\). Otherwise, the writer needs to liquidate his holding of stock, so \(y_T = 0\). This amounts to imposing the following terminal constraints on the processes \((L, M)\):

\[
dL_T = (\Delta(T, S_T) - y)_+ \quad \text{and} \quad dM_T = (y - \Delta(T, S_T))_+,
\]

where we write \(y\) for the stock holding just prior to expiration. With this formulation, we recover the Black-Scholes hedging strategy \((y_t = \Delta(t, S_t))\) when \(\lambda = \mu = 0\).
2.1 A Free Boundary Problem and Its Connection to Optimal Stopping

Consider the stochastic control problem whose value function at time $t$ (i.e., with $T - t$ units of time left) is

\begin{equation}
V(t, S, y) = \inf_{L,M} J(t, S, y), \quad (t, S, y) \in [0, T] \times (0, \infty) \times \mathbb{R},
\end{equation}

when the current stock price is $S$ and stock holding is $y$. We can obtain key insights into the nature of the optimal hedging strategy by temporarily restricting $L$ and $M$ in (2.6) to be absolutely continuous processes with positive derivatives uniformly bounded by $\kappa$, i.e.,

$$L_t = \int_0^t \ell_u \, du \quad \text{and} \quad M_t = \int_0^t m_u \, du, \quad 0 \leq \ell_u, m_u \leq \kappa < \infty.$$ 

Thus, the value function becomes

$$V(t, S, y) = \inf_{L,M} E \left\{ \int_t^T [F(u, S_u, y_u) + (\lambda \ell_u + \mu m_u)S_u/K] \, du \mid S_t = S, y_t = y \right\}$$

and the infinitesimal generator of the stochastic system comprising (2.1) and $dy_t = (\ell_t - m_t) \, dt$ is

$$\mathcal{L} = (\alpha - q)S \frac{\partial}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (\ell - m) \frac{\partial}{\partial y}.$$ 

The Bellman equation for the value function is

$$\min_{\ell,m} \{ (\partial/\partial t + \mathcal{L})V(t, S, y) + F(t, S, y) + (\lambda \ell + \mu m)S/K \} = 0,$$

or equivalently,

\begin{equation}
\min_{\ell,m} \{ (\lambda S/K + \partial V/\partial y)\ell + (\mu S/K - \partial V/\partial y)m \}
\end{equation}

$$+ (\alpha - q)S\partial V/\partial S + (\sigma^2 S^2/2)\partial^2 V/\partial S^2 + \partial V/\partial t + F = 0.$$

Since the value function is increasing in $|y - \Delta(t, S)|$ and therefore $\partial V/\partial y < 0$ if $y < \Delta(t, S)$ and $\partial V/\partial y > 0$ if $y > \Delta(t, S)$, it follows that the minimum in the left-hand side of (2.7) is attained by

(i) buying at the maximum rate $\ell = \kappa$ and $m = 0$ when $y < \Delta(t, S)$ and $\partial V/\partial y \leq -\lambda S/K$;

(ii) selling at the maximum rate $m = \kappa$ and $\ell = 0$ when $y > \Delta(t, S)$ and $\partial V/\partial y \geq \mu S/K$;

(iii) doing nothing: $\ell = m = 0$ when $-\lambda S/K \leq \partial V/\partial y \leq \mu S/K$.

Thus, the state space $[0, T] \times (0, \infty) \times \mathbb{R}$ is partitioned into (i) a “buy stock” region, (ii) a “sell stock” region, and (iii) a “no transaction” region, respectively. For sufficiently large $\kappa$, $V(t, S, y) = V(t, S, y + \kappa)$.
\( \delta y + \lambda \delta y S/K \) in the buy region (with \( \delta y \) shares of stock bought) and \( V(t, S, y) = V(t, S, y - \delta y) + \mu \delta y S/K \) in the sell region (with \( \delta y \) shares of stock sold). Instantaneous transaction from the interior of the buy (resp. sell) region to the buy (resp. sell) boundary takes place by letting \( \kappa \to \infty \), yielding the following free boundary problem for the singular stochastic control value function (2.6):

\[
\begin{align*}
(\alpha - q) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + F &= 0 \quad \text{in } \mathcal{N}, \\
\frac{\partial V}{\partial y}(t, S, y) &= -\lambda S/K \quad \text{in } \partial \mathcal{N} \cap \{y < \Delta(t, S)\}, \\
\frac{\partial V}{\partial y}(t, S, y) &= \mu S/K \quad \text{in } \partial \mathcal{N} \cap \{y > \Delta(t, S)\},
\end{align*}
\]

where \( \mathcal{N} \) is the no-transaction region; see similar arguments of Karatzas (1983) and Davis et al. (1993).

With \( \beta = \alpha / \sigma^2 \) and \( \theta = q / \alpha \), a more parsimonious parameterization of problem (2.8) can be obtained by considering the transformations

\[
\begin{align*}
s &= \sigma^2 (t - T), \quad z = \log(S/K) - (\beta - \theta - 1/2) s, \quad x = y, \quad v(s, z, x) &= V(t, S, y),
\end{align*}
\]

Applying the chain rule of differentiation yields

\[
\begin{align*}
\frac{\partial V}{\partial S} &= \frac{1}{S} \frac{\partial v}{\partial z}, \quad \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2} \left( \frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z} \right), \quad \frac{\partial V}{\partial t} = \sigma^2 \frac{\partial v}{\partial s} - \left( \alpha - q - \frac{\sigma^2}{2} \right) \frac{\partial v}{\partial z},
\end{align*}
\]

which, upon substitution into problem (2.8), results in the free boundary problem

\[
\begin{align*}
\frac{1}{2} \frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial s} + f &= 0 \quad \text{in } \mathcal{N}, \\
\frac{\partial v}{\partial x}(s, z, x) &= -\lambda e^{z + (\beta - \theta - 1/2) s} \quad \text{in } \partial \mathcal{N} \cap \{x < D(s, z)\}, \\
\frac{\partial v}{\partial x}(s, z, x) &= \mu e^{z + (\beta - \theta - 1/2) s} \quad \text{in } \partial \mathcal{N} \cap \{x > D(s, z)\},
\end{align*}
\]

where

\[
D(s, z) = \Delta(t, S), \quad f(s, z, x) = F(t, S, y) / \sigma^2 = e^{2z + (2\beta(1-\theta)-1)s}[x - D(s, z)]^2.
\]

Arguing formally as in Bather and Chernoff (1967) who studied the problem of controlling the motion of a spaceship relative to its target on a finite horizon with an infinite amount of fuel, we observe that problem (2.10) can be simplified by working with

\[
u(s, z, x) = \frac{\partial v}{\partial x}(s, z, x),
\]

thereby reducing the gradient constraints (2.10b) and (2.10c) to constraints on the newly defined value function. Letting

\[
g(s, z, x) = \partial f(s, z, x) / \partial x = 2e^{2z + (2\beta(1-\theta)-1)s}[x - D(s, z)],
\]

\[
G(s, z, x) = [\mu\mathbb{I}_{\{x > D(s, z)\}} - \lambda\mathbb{I}_{\{x < D(s, z)\}}]e^{z + (\beta - \theta - 1/2) s},
\]

\[
\frac{1}{2} \frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial s} + f = 0 \quad \text{in } \mathcal{N}, \\
\frac{\partial v}{\partial x}(s, z, x) &= -\lambda e^{z + (\beta - \theta - 1/2) s} \quad \text{in } \partial \mathcal{N} \cap \{x < D(s, z)\}, \\
\frac{\partial v}{\partial x}(s, z, x) &= \mu e^{z + (\beta - \theta - 1/2) s} \quad \text{in } \partial \mathcal{N} \cap \{x > D(s, z)\},
\]

where

\[
(2.11)
\]

\[
D(s, z) = \Delta(t, S), \quad f(s, z, x) = F(t, S, y) / \sigma^2 = e^{2z + (2\beta(1-\theta)-1)s}[x - D(s, z)]^2.
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Arguing formally as in Bather and Chernoff (1967) who studied the problem of controlling the motion of a spaceship relative to its target on a finite horizon with an infinite amount of fuel, we observe that problem (2.10) can be simplified by working with

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\]

\[
G(s, z, x) = [\mu\mathbb{I}_{\{x > D(s, z)\}} - \lambda\mathbb{I}_{\{x < D(s, z)\}}]e^{z + (\beta - \theta - 1/2) s},
\]
note that \( u \) is a solution of the free boundary problem
\[
\begin{align*}
(2.14a) \quad & \frac{1}{2} \frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial s} + g = 0 \quad \text{in} \ N, \\
(2.14b) \quad & u(s, z, x) = -\lambda e^{z+(\beta-\theta\beta-1/2)s} \quad \text{in} \ \partial N \cap \{x < D(s, z)\}, \\
(2.14c) \quad & u(s, z, x) = \mu e^{z+(\beta-\theta\beta-1/2)s} \quad \text{in} \ \partial N \cap \{x > D(s, z)\}.
\end{align*}
\]

We next show that \( u(s, z, x) \) is the value function of an optimal stopping problem for standard Brownian motion \( Z_t \):
\[
(2.15) \quad u(s, z, x) = \begin{cases} 
\inf_{\tau \in T(s, 0)} E \left\{ \int_s^\tau g(u, Z_u, x_u) \, du + G(\tau, Z_\tau, x_\tau) \bigg| Z_s = z, x_s = x \right\} & \text{if } x > D(s, z), \\
\sup_{\tau \in T(s, 0)} E \left\{ \int_s^\tau g(u, Z_u, x_u) \, du + G(\tau, Z_\tau, x_\tau) \bigg| Z_s = z, x_s = x \right\} & \text{if } x < D(s, z),
\end{cases}
\]
\((s, z, x) \in [-\sigma^2 T, 0] \times \mathbb{R} \times \mathbb{R}, \) where \( T(a, b) \) denotes the set of stopping times taking values between \( a \) and \( b \) (\( b > a \)). Moreover, the optimal continuation region of problem (2.15) (a restatement of problem (2.14)) coincides with the no-transaction region of the singular stochastic control problem (2.10). Such an “equivalence” between singular stochastic control and optimal stopping has been rigorously established by Karatzas and Shreve (1985) for the reflected follower problem of choosing a control process of bounded variation, subject to a reflecting barrier at the origin, to minimize an expected cost. A summary of this equivalence theory is given in the Appendix. Problem (2.10), written in the form
\[
(2.16) \quad v(s, z, x) = \inf_{L, M} E \left\{ \int_s^0 f(u, Z_u, x_u) \, du \\
+ \int_s^0 e^{Z_u+(\beta-\theta\beta-1/2)u} \left( \lambda dL_u + \mu dM_u \right) \bigg| Z_s = z, x_s = x \right\},
\]
is a special case of the reflected follower problem, except that the control \( x = L - M \) in problem (2.16) is not applied directly to the Brownian motion and that the problem is not symmetric about \( x = D(s, z) \). Nevertheless, similar arguments as in Karatzas and Shreve (1985) can be used to prove that \( v(s, z, x) \) given by (2.16) and \( u(s, z, x) \) given by (2.15) are indeed related by (2.12), and that the optimal continuation region of problem (2.15) is the same as the no-transaction region of problem (2.16) (a restatement of problem (2.10)). Since \( u(s, z, x) \) is nondecreasing in \( x \), there exist sell and buy boundaries, denoted respectively by \( X_s(s, z) \) and \( X_b(s, z) \), such that \( X_s(s, z) \) is the smallest \( x > D(s, z) \) for which \( u(s, z, x) = \mu e^{x+(\beta-\theta\beta-1/2)s} \) and \( X_b(s, z) \) is the largest \( x < D(s, z) \) for which \( u(s, z, x) = -\lambda e^{x+(\beta-\theta\beta-1/2)s} \). Thus, if \( x > X_s(s, z) \) (resp. \( x < X_b(s, z) \)), the option writer must immediately sell \( x - X_s(s, z) \) (resp. buy \( X_b(s, z) - x \)) shares of stock to form an optimal hedge.
2.2 A Backward Induction Algorithm to Solve for the Buy/Sell Boundaries

Since the horizon of the optimal stopping problem (2.15) is always zero, only one numerical program for each set of parameter values has to be implemented for all expiration dates $T$. In view of the functional central limit theorem, we can approximate standard Brownian motion by a symmetric Bernoulli random walk. Thus, taking a small $\delta > 0$, a numerical solution to the optimal stopping problem (2.15) is provided by the following backward induction algorithm for the approximating random walk:

\[
(2.17) \quad u(s_i, z, x) = \begin{cases} 
\min\{\mu e^{z+(\beta-\theta-1/2)s_i}, \tilde{u}(s_i, z, x)\} & \text{if } x > D(s_i, z), \\
\max\{-\lambda e^{z+(\beta-\theta-1/2)s_i}, \tilde{u}(s_i, z, x)\} & \text{if } x < D(s_i, z),
\end{cases}
\]

where $z \in Z_\delta = \{0, \pm \sqrt{\delta}, \pm 2\sqrt{\delta}, \ldots\}$, $s_0 = 0$, $s_i = s_{i-1} - \delta$, and

\[
(2.18) \quad \tilde{u}(s, z, x) = \delta g(s, z, x) + [u(s + \delta, z + \sqrt{\delta}, x) + u(s + \delta, z - \sqrt{\delta}, x)]/2.
\]

In view of the terminal constraints (2.5), the backward induction algorithm (2.17) is initialized with

\[
(2.19) \quad u(0, z, x) = \mu e^{z}1_{\{x > D(0, z)\}} - \lambda e^{z}1_{\{x < D(0, z)\}}.
\]

At time $s_i$, each point $(z, x) \in Z_\delta \times \mathbb{R}$ can be classified as belonging to the sell region, buy region, or no-transaction region, depending on whether $u(s_i, z, x) = \mu e^{z+(\beta-\theta-1/2)s_i}$, $u(s_i, z, x) = -\lambda e^{z+(\beta-\theta-1/2)s_i}$, or $-\lambda e^{z+(\beta-\theta-1/2)s_i} < u(s_i, z, x) < \mu e^{z+(\beta-\theta-1/2)s_i}$, respectively. If $X^\delta_s(s_i, z)$ is the stopping point in the sell region closest to the no-transaction region, we approximate the continuous-time sell boundary $X_s(s_i, z)$ by linearly interpolating $\tilde{u}(s_i, z, X^\delta_s(s_i, z) - \delta)$ and $\tilde{u}(s_i, z, X^\delta_s(s_i, z))$ to solve for that $x \in [X^\delta_s(s_i, z) - \delta, X^\delta_s(s_i, z)]$ for which $\tilde{u}(s_i, z, x) = \mu e^{z+(\beta-\theta-1/2)s_i}$. For the buy boundary $X^\delta_b(s_i, z)$, we first locate $X^\delta_b(s_i, z)$, the stopping point in the buy region closest to the no-transaction region, and then linearly interpolate $\tilde{u}(s_i, z, x)$ between $X^\delta_b(s_i, z)$ and $X^\delta_b(s_i, z) + \delta$ to solve $\tilde{u}(s_i, z, x) = -\lambda e^{z+(\beta-\theta-1/2)s_i}$.

The hedging problem of the option buyer is analogous to that of the option writer, with straightforward modifications due to the fact that the hedging portfolio at time $t$ now consists of $+1$ option and $y_t$ shares of stock. Specifically, for a long position in the option, we take $F(t, S, y) = \sigma^2(S/K)^2[y + \Delta(t, S)]^2$ in (2.4) and $g(s, z, x) = 2e^{2z+(2\beta(1-\theta)-1)s}[x + D(s, z)]$ in (2.14a), and carry out the following analog of (2.17):

\[
(2.20) \quad u(s_i, z, x) = \begin{cases} 
\min\{\mu e^{z+(\beta-\theta-1/2)s_i}, \tilde{u}(s_i, z, x)\} & \text{if } x > -D(s_i, z), \\
\max\{-\lambda e^{z+(\beta-\theta-1/2)s_i}, \tilde{u}(s_i, z, x)\} & \text{if } x < -D(s_i, z),
\end{cases}
\]

where $\tilde{u}(s, z, x)$ is given by (2.18) with $g(s, z, x) = 2e^{2z+(2\beta(1-\theta)-1)s}[x + D(s, z)]$. It follows from equation (2.20) that $-u(s_i, z, -x)$ has exactly the same form as the right-hand side of (2.17) with $\lambda$ and $\mu$
interchanged. This means that the backward induction algorithm for the option writer can also be used to solve for the optimal boundaries of the option buyer. For an option buyer with proportional sell rate \( \mu \) and proportional buy rate \( \lambda \), the optimal sell and buy boundaries, denoted respectively by \( \tilde{X}_s(s, z) \) and \( \tilde{X}_b(s, z) \), can be obtained via the relations \( \tilde{X}_s(s, z) = -X_b(s, z) \) and \( \tilde{X}_b(s, z) = -X_s(s, z) \), where \( X_s(s, z) \) and \( X_b(s, z) \) are the optimal sell and buy boundaries for an option writer with proportional rates interchanged: sell rate \( \lambda \) and buy rate \( \mu \).

2.3 The Option’s Market Price and Its Delta

It is customary to regard the Black-Scholes price of an option as the market price. Let

\[
d_1(t, S) = \frac{\log(S/K) + (r - q)(T - t)}{\sigma \sqrt{T - t}} + \frac{\sigma \sqrt{T - t}}{2}
\]

and

\[
d_2(t, S) = d_1(t, S) - \sigma \sqrt{T - t},
\]

where \( n(x) = e^{-x^2/2}/\sqrt{2\pi} \) and \( N(x) = \int_{-\infty}^{x} n(t) \, dt \) denote the standard normal density and distribution functions, respectively. Taking the Black-Scholes price

\[
P(t, S) = \pm \{ Se^{-q(T-t)} N(\pm d_1(t, S)) - Ke^{-r(T-t)} N(\pm d_2(t, S)) \},
\]

the delta in (2.3) is given by

\[
\Delta(t, S) = \partial P(t, S)/\partial S = \pm e^{-q(T-t)} N(\pm d_1(t, S)).
\]

In its limiting form as \( t \to T \), \( \Delta(T, S) = \pm 1_{\{ (S-K)>0 \}} \). With the choice of (2.22) as the option’s market price, we use

\[
D(s, z) = \pm e^{\theta \beta s} N(\pm \{ z/\sqrt{-s} + (\rho - \beta + 1)\sqrt{-s} \})
\]

in the backward induction (2.17), where \( \rho = r/\sigma^2 \). As an illustration, optimal sell and buy boundaries of the call option writer for \( \beta = \rho = 0.5, \theta = 0 \), and \( \lambda = \mu = 0.005 \) are shown in Figure 2.1. In Figure 2.2, we compare the optimal sell and buy boundaries for various values of proportional rates: \( \lambda = \mu = 0.05, 0.005, 0.0005, 0 \) (for which both boundaries coincide with the Black-Scholes delta \( D(s, z) \)). Note that the proportional transaction cost rates determine a monotone sequence of sell boundaries (increasing as sell rates increase) and buy boundaries (decreasing as buy rates increase).

INSERT FIGURES 2.1 AND 2.2 ABOUT HERE

Since transaction costs are assumed to be positive in our optimal hedging theory, one may argue that this violation of the Black-Scholes assumptions invalidates the use of (2.22) as the option price. In practice
there are also other violations of the Black-Scholes assumptions, and an alternative to adopting the Black-Scholes value as the market price is to use a nonparametric method for estimating the pricing formula of the option. Then, instead of the Black-Scholes delta, we would set \( \Delta(t, S) = \partial P_{\text{np}}(t, S)/\partial S \), where \( P_{\text{np}} \) is the nonparametric pricing formula. Such an approach was first proposed by Hutchinson et al. (1994) (in the absence of transaction costs) who used learning networks, namely, neural networks, or radial basis function or projection pursuit regression networks, for nonparametric pricing. Because the estimation of parameters in such networks involves computationally intensive nonlinear optimization procedures, Lai and Wong (2004) and Lai and Lim (2004) propose to use univariate regression splines and their tensor products as basis functions of the network. An empirical study, using the daily settlement prices of S&P 500 futures and futures options, of nonparametric option pricing and its application to optimal hedging in the presence of transaction costs is given in Lai and Lim (2004).

3. UPPER AND LOWER OPTION VALUES

Section 2 formulates a *penalized* risk-minimization problem that determines optimal hedging strategies for both long and short positions in an option. It also provides a backward induction algorithm to solve for the optimal sell and buy boundaries (in the transformed \((s, z)\)-coordinates) associated with the optimal strategies. Let \( Y_s(t, S) = X_s(s, z) \), \( Y_b(t, S) = X_b(s, z) \) (for the option writer) and \( \tilde{Y}_s(t, S) = \tilde{X}_s(s, z) \), \( \tilde{Y}_b(t, S) = \tilde{X}_b(s, z) \) (for the option buyer) be the optimal boundaries in the original \((t, S)\)-coordinates.

Then, the optimal holding \( y^* \) of shares is given by

\[
y^*_t = \begin{cases} 
Y_b(t, S_t) & \text{if } y^*_{t-} < Y_b(t, S_t), \\
Y_s(t, S_t) & \text{if } y^*_{t-} > Y_s(t, S_t), \\
\tilde{y}^*_{t-} & \text{if } Y_b(t, S_t) \leq y^*_{t-} \leq \tilde{Y}_s(t, S_t),
\end{cases}
\]

for the option writer, with \( Y_s \) and \( Y_b \) replaced by \( \tilde{Y}_s \) and \( \tilde{Y}_b \) for the option buyer. The first (resp. second) case in (3.1) involves an immediate purchase (resp. sale) of shares of stock in order to rebalance from \( y^*_{t-} \) to \( Y_b(t, S_t) \) (resp. \( Y_s(t, S_t) \)). With \( y^*_{0-} = 0 \) (zero initial holding of stock), the corresponding hedging cost at time \( t \) is \( H(t, S_t, y^*_{t-}) \), where

\[
H(t, S, y) = S \left[ (1 + \lambda)(Y_b(t, S) - y)_+ - (1 - \mu)(y - Y_s(t, S))_+ \right]
\]

for the option writer, with \( Y_s \) and \( Y_b \) replaced by \( \tilde{Y}_s \) and \( \tilde{Y}_b \) for the option buyer. We define option values for both the writer and the buyer as expected total hedging costs resulting from following their respec-
tive optimal hedging strategies. These definitions lead to upper (corresponding to the writer’s) and lower (corresponding to the buyer’s) bounds for the option price in the presence of transaction costs.

3.1 Short Position in Option

We will make use of a self-financing argument to derive the option value when the option writer follows the optimal trading strategy given by (3.1), which necessitates an instantaneous change in the amount of shares whenever the holding of shares moves outside the allowable range, and maintains a risk-free account to finance the trades. Suppose that the first trade after time \( t \) occurs at time \( \tau \), so \( y_t^* = y_t^\tau \) for all \( u \in [t, \tau) \) and \( y_\tau^* = Y_b(\tau, S_\tau) \) or \( Y_s(\tau, S_\tau) \) according as whether it is optimal to buy or sell stock at \( \tau \). This trade at \( \tau \) incurs a cost of \( H(\tau, S_\tau, y_\tau^\tau) \). In the time that has elapsed, the risk-free account grows (in absolute value) from \( B_t \) to \( B_t e^{r(\tau - t)} \) and the stock holding pays out a dividend amount of \( q \int_t^\tau y_u^* S_u \, du \). In order for the rebalancing trade at time \( \tau \) to be self-financing, the following condition needs to be satisfied by the dividend payout, the hedging cost, the new value \( B_\tau \) in the risk-free account, and the old value in the risk-free account:

\[
B_t e^{r(\tau - t)} + q \int_t^\tau y_u^* S_u \, du = H(\tau, S_\tau, y_\tau^\tau) + B_\tau.
\]

We can rewrite this condition as

\[
B_t = e^{-r(\tau-t)} [H(\tau, S_\tau, y_\tau^\tau) - q \int_t^\tau y_u^* S_u \, du] + e^{-r(\tau-t)} B_\tau.
\]

More generally, suppose trades after time \( t (= \tau_0) \) occur at fixed times \( \tau_1 < \tau_2 < \cdots < \tau_n = T \). The preceding discussion shows that

\[
B_{\tau_{i-1}} = e^{-r(\tau_{i-1}-\tau_{i-1})} \left[ H(\tau_i, S_{\tau_i}, y_{\tau_i}^\tau) - q \int_{\tau_{i-1}}^{\tau_i} y_u^* S_u \, du \right] + e^{-r(\tau_{i-1}-\tau_{i-1})} B_{\tau_i}, \quad i = 1, \ldots, n,
\]

from which it follows (by induction) that the amount of (initial) capital required at time \( t \) to finance the downstream trades is

\[
B_t = \sum_{i=1}^{n} e^{-r(\tau_{i-1}-\tau_{i-1})} \left[ H(\tau_i, S_{\tau_i}, y_{\tau_i}^\tau) - q \int_{\tau_{i-1}}^{\tau_i} y_u^* S_u \, du \right] + e^{-r(T-t)} B_T,
\]

subject to the following terminal (option exercise and settlement) conditions:

\[
y_T^* = \pm \mathbb{I}_{\{\pm S_T > \pm K\}} \quad \text{and} \quad B_T = \mp K \mathbb{I}_{\{y_T^* = \pm 1\}}.
\]

Note that \( H(\tau, S_\tau, y_\tau^\tau) = 0 \) if no trade occurs at time \( \tau \). Replacing the sum in the right-hand side of (3.3) by an integral and taking expectation to handle random times at which trades occur, the option writer would therefore ask for the following compensation to take on the risk of writing an option:

\[
P(t, S, y) = E \left\{ \int_t^T e^{-r(\tau-t)} [H(\tau, S_\tau, y_\tau^\tau) - q y_\tau^* S_\tau] \, d\tau \ \bigg| \ S_t = S, \ y_t^\tau = y \right\}
\]

\[
\mp K e^{-r(T-t)} \ Pr\{y_T^* = \pm 1 \mid S_t = S, \ y_t^\tau = y\}.
\]
The initial cost of writing the option is given by (3.5) with the convention that \( y_{0-} = 0 \) (i.e., the option writer owns zero shares of stock at time 0). Note that in contrast to Boyle and Vorst (1992), we assume that the option writer creating the replicating portfolio has to buy an initial amount of stock if it is optimal to do so. An upper bound for the option writer’s price is

\[
\mathcal{P}(t, S) = \max\{ P(t, S, y) : y \in [Y_b(t-, S_{t-}), Y_s(t-, S_{t-})]\},
\]

in which the maximization arises from the fact that if the optimal hedging strategy is followed, the option writer creating the replicating portfolio can end up at time \( t- \) with an amount of stock in the interval \([Y_b(t-, S_{t-}), Y_s(t-, S_{t-})]\). The maximization in (3.6) can be performed by grid search.

Since the computations in Section 2 have been carried out in the transformed coordinates (2.9), we rewrite (3.5) accordingly. With \( u = \sigma^2(\tau - T), r = \sigma^2(t - T) \) and \( x_u^* = y_u^* \), the writer’s option value can be written as

\[
p(s, z, x) = Ke^{rS} \left\{ \int_{s}^{0} e^{-\rho u} [h(u, Z_u, x_u^*) - \theta \beta x_u^* e^{Z_u + (\beta - \theta \beta - 1/2)u}] \, du \bigg| Z_s = z, x_{s-}^* = x \right\} \\
+ \Pr\{ x_0^* = \pm 1 \mid Z_s = z, x_{s-}^* = x \},
\]

where \( Z_u \) is a standard Brownian motion and \( h(s, z, x) = e^{z+ (\beta - \theta \beta - 1/2)s} [h_b(s, z, x) - h_s(s, z, x)] \), with

\[
h_b(s, z, x) = (1 + \lambda)(X_b(s, z) - x)_{+} \quad \text{and} \quad h_s(s, z, x) = (1 - \mu)(x - X_s(s, z))_{+}.
\]

The upper bound (3.6) can be expressed as the maximum of (3.7) over the interval \([X_b(s-, z_{s-}), X_s(s-, z_{s-})]\). Moreover, the event \( \{ x_0^* = \pm 1 \} \) in (3.7) is equivalent to the event that the option expires in the money, so

\[
\Pr\{ x_0^* = \pm 1 \mid Z_s = z, x_{s-}^* = x \} = N(\pm z/\sqrt{s}).
\]

For small \( \delta \) such that \( m = -s/\delta \) is an integer, we can approximate the integral \( \int_{s}^{0} e^{-\rho u} h(u, Z_u, x_u^*) \, du \) by the sum \( I_m = \sum_{i=0}^{m} e^{i\rho \delta} h(-i\delta, Z_{-i\delta}, x_{-i\delta+1}^*) \). With this discrete-time approximation, the writer’s option value (3.7) can be computed by the following Monte Carlo procedure, which has the advantage that it can also be used to simulate the probability distributions of trading frequency and average trade size (see Section 4). The procedure consists of running a large number \( N \), say \( N \) replicates of the sum \( I_m \) and then averaging over these \( N \) replicates to produce an estimate for the expectation of the sum. In each simulation, we first generate a sequence of independent zero-mean normal random variables \( \zeta_1, \ldots, \zeta_m \), each with variance \( \delta \), to produce the discrete-time Brownian path \( Z_{-i\delta} = Z_{-(i+1)\delta} + \zeta_i \) \( (i = 0, 1, \ldots, m - 1) \) with \( Z_{-m\delta} = z \). Next, starting with \( x_{-(m+1)\delta}^* = x \), we check for \( i = m, m - 1, \ldots, 0 \) whether it is optimal to adjust the holding in stock, and compute the rebalancing cost if it is optimal to do so:
Analogous to (3.6), a

\[ P \text{ with } y \]

\[ (3.10) \]

\[ y \]

\[ (3.9) \]

\[ I \]

\[ \text{translation run can also be used to compute the other integral} \]

\[ \text{Analogously, for a long position in the option, the option buyer would have to pay} \]

\[ \text{estimates and their standard errors are given in Table 3.1.} \]

\[ \text{option values, Table 3.1 also gives the Monte Carlo estimates of the lower option values and their standard} \]

\[ \text{sum is estimated. The probability of the option expiring in the money (the last term of (3.7)) is similarly} \]

\[ \text{estimated by recoding the terminal states as 1 (in the money) or 0 (at or out of the money) and then averaging} \]

\[ \text{over these 0-1 values given by} \]

\[ \text{Using } m = 2,000 \text{ and } N = 10,000 \text{ simulation paths, we compute the upper option values (3.7) for} \]

\[ S = 100, \alpha = r = 0.08, \sigma = 0.4, T - t = 1.25, \text{ and various values of } K \text{ and } \lambda = \mu. \]

\[ \text{Both the Monte Carlo estimates and their standard errors are given in Table 3.1.} \]

\[ \text{INSERT TABLE 3.1 ABOUT HERE} \]

3.2 Long Position in Option

Analogously, for a long position in the option, the option buyer would have to pay

\[ (3.9) \]

\[ P(t, S, y) = - \left[ E \left\{ \int_t^T e^{-r(t-\tau)} [H(\tau, S_\tau, y^*_\tau) - qy^*_\tau S_\tau] \, d\tau \bigg| S_t = S, y^*_{t-} = y \right\} \right. \]

\[ \left. \pm Ke^{-r(T-t)} \Pr \{ y^*_T = \pm 1 \mid S_t = S, y^*_{t-} = y \} \right] , \]

with \( y^*_{0-} = 0 \), cf. (3.5). Note that the terminal conditions in the self-financing argument for this case are

\[ (3.10) \]

\[ y^*_{T+} = \mp \Pi_{\{ \pm S_T > \pm K \}} \quad \text{and} \quad B_T = \pm K \Pi_{\{ y^*_{T-} = \mp 1 \}} . \]

Analogous to (3.6), a lower bound for the option buyer’s price is

\[ (3.11) \]

\[ \bar{P}(t, S) = \min \{ P(t, S, y) : y \in [\tilde{Y}_b(t-, S_{t-}), \tilde{Y}_s(t-, S_{t-})] \} . \]

where \( \tilde{Y}_b \) and \( \tilde{Y}_s \) denote the buy and sell boundaries, respectively, of the option buyer. Besides the upper option values, Table 3.1 also gives the Monte Carlo estimates of the lower option values and their standard errors.
4. PROPERTIES OF OPTIMAL HEDGE

In this section, we study numerically some typical behavior of our optimal hedging strategy, and compare its performance with that of the discrete-time replicating strategy of Leland (1985) and that of the utility-based hedging strategy of Clewlow and Hodges (1997).

4.1 Transaction Frequency and Average Trade Size

When the simulation procedure described in Section 3 was used to compute the upper and lower option values reported in Table 3.1, 10,000 stock price paths were simulated and tracked to determine the optimal holding of shares for each simulated path. For the parameters used in Table 3.1, typical optimal holdings of shares over time are shown in Figure 4.1 for stock price paths that moved “down”, were “flat”, or moved “up.” Encoded in the optimal holding of shares are the times and sizes of trades, from which we can compute the transaction frequency (ratio of number of optimal trades to total number of trade opportunities, the latter being 2,000 in this illustration) and average trade size (ratio of total number of traded shares to number of optimal trades). Figure 4.2 shows that transaction frequency tends to decrease as proportional transaction cost rates increase, since it becomes more expensive to rebalance too often. This in turn results in the actual hedge ratio falling much farther out of line from the “target” hedge ratio before it becomes necessary to rebalance, so the average trade size tends to increase with proportional transaction cost rates (see Figure 4.3).

INSERT FIGURES 4.1, 4.2, AND 4.3 ABOUT HERE

4.2 Comparison with Discrete-Time and Utility-Based Hedges

To compare our optimal hedging strategy with the discrete-time replicating strategy of Leland (1985) and the utility-based approach of Clewlow and Hodges (1997), we compute the mean and standard deviation of the hedging error (see the next paragraph) of each strategy, over a set of values of a tuning parameter to plot the mean versus standard deviation curve. We also include in our comparison the discrete-time Black-Scholes strategy based on an unadjusted volatility. For the Black-Scholes and Leland strategies, the tuning parameter is the revision interval $\delta t$ (which is varied from 1 to 12 days). For the Clewlow-Hodges strategy, the tuning parameter is the risk aversion coefficient $\gamma$ of the negative exponential utility function $U(w) = -e^{-\gamma w}$ and is varied from 0.2 to 10. Similarly, for our risk minimization strategy, we also introduce a risk aversion-type coefficient $\gamma$ to $\int_t^T F(u, S_u, y_u) \, du$ in (2.4) and vary it from 5 to 100 as a tuning parameter to obtain the
mean-variance profile. For a given value of $\gamma$, the backward induction algorithm (2.17) with $(\lambda, \mu)$ replaced by $(\lambda/\gamma, \mu/\gamma)$ is used to compute the optimal hedging strategy.

We follow Clewlow and Hodges (1997) to define the hedging error of a hedging strategy. Specifically, we consider selling and buying an option for the Black-Scholes fair value (without transaction costs) and using the proceeds to replicate the option position under transaction costs. For each strategy, we compute the value of the replicating portfolio (stock and cash less liability) at maturity and discount this value back to the present to form the hedging error; e.g. (3.3) minus the Black-Scholes value, with terminal conditions (3.4) for a short call or (3.10) for a long call. We use $N = 10,000$ simulated stock price paths to estimate the mean and standard deviation of this hedging error. The Black-Scholes and Leland strategies require the stock holding to be adjusted at each revision date to the Black-Scholes delta, with an unadjusted volatility $\sigma$ and with an adjusted volatility $\tilde{\sigma} = \sigma [1 + (\lambda + \mu) \sqrt{2/\pi/\sigma \sqrt{\delta t}}]^{1/2}$, respectively. With $n = T/\delta t$ an integer, we set $y^*_t = \Delta(\tau_i, S_{\tau_i})$ for $\tau_i \leq t < \tau_{i+1}$ with $\tau_i = i\delta t$ ($0 \leq i \leq n$) and $Y_s(t, S) = Y_b(t, S) = \Delta(t, S)$ for a short call (or $\tilde{Y}_s(t, S) = \tilde{Y}_b(t, S) = -\Delta(t, S)$ for a long call) in (3.2) to compute $N$ replicates of (3.3) minus the unadjusted Black-Scholes value, from which the mean and standard deviation of the hedging error can be evaluated. This is repeated for various values of $\delta t$. For the Clewlow-Hodges strategy and our risk minimization strategy, the optimal stock holding limits ($Y_s$ and $Y_b$ for a short call, or $\tilde{Y}_s$ and $\tilde{Y}_b$ for a long call) are first computed on a fine $(t, S)$-grid for various values of $\gamma$ and stored. These stored limits are then used with (3.1) and the simulated stock price paths to determine, at each $t$ on the grid, whether it is necessary to adjust the stock holding. The hedging error is then obtained using (3.3) less the unadjusted Black-Scholes value, from which the mean and standard deviation of the hedging error can again be evaluated. As Figure 4.4 shows, our optimal hedging strategy outperforms the other three strategies in the mean-variance sense: for a given mean hedging error, our strategy has the smallest standard deviation (i.e., is least risky).

5. EXTENSIONS

Our treatment in Section 2 has focused on transaction fees proportional to the value of shares traded, on settlement by physical delivery of a single share of stock, and on European options. We now extend our formulation to include fees proportional to the number of shares traded and settlement by cash, which can still be handled by using the methodology of Sections 2 and 3. The case of American options, for which early exercise is permissible, is treated by Lai and Lim (2004).
5.1 Transaction Fees Proportional to Number of Shares Traded

Suppose that in addition to the proportional cost rates $\lambda$ and $\mu$ (which we now rewrite as $\lambda_1$ and $\mu_1$) considered in Section 2, it costs $\lambda_0$ (resp. $\mu_0$) per share bought (resp. sold). Then the criterion (2.4) becomes

\[
J(t, S, y) = E \left\{ \int_t^T F(u, S_u, y_u) \, du + \int_t^T [\lambda_0 + \lambda_1(S_u/K)] \, dL_u + \int_t^T [\mu_0 + \mu_1(S_u/K)] \, dM_u \right\}.
\]

Following the arguments in Section 2, the backward induction algorithm (2.17) initialized with (2.19) can be modified as follows:

\[
u(s_i, z, x) = \begin{cases} \min\left\{ \mu_0 + \mu_1 e^{z \bar{\beta}(s_i, z, x)} \right\} & \text{if } x > D(s_i, z), \\ \max\left\{ -\lambda_0 - \lambda_1 e^{z \bar{\beta}(s_i, z, x)} \right\} & \text{if } x < D(s_i, z), \end{cases}
\]

with $\bar{\nu}(s_i, z, x)$ given by (2.18) and

\[
u(0, z, x) = (\mu_0 + \mu_1 e^z)\mathbb{I}_{\{x > D(0, z)\}} - (\lambda_0 + \lambda_1 e^z)\mathbb{I}_{\{x < D(0, z)\}}.
\]

Once the optimal hedging strategy has been solved for, the expected hedging costs can be computed by using the procedure described in Section 3 with

\[
H(t, S, y) = [\lambda_0 + (1 + \lambda_1)S](Y_b(t, S) - y)_+ + [\mu_0 - (1 - \mu_1)S](y - Y_s(t, S))_+
\]

for the option writer, where $Y_b$ and $Y_s$ are replaced by $\tilde{Y_b}$ and $\tilde{Y_s}$ for the option buyer.

5.2 Option Settlement by Cash

The formulation in Section 2 assumes that the option is settled by physical delivery, as is the case for options on equities (stocks and funds), futures, and bonds. In contrast, if the underlying is a currency, fixed-income instrument, stock index (or basket of stocks), or a stock of restricted transferability, cash settlement of the option is the norm. In this case, the option writer always liquidates his holding of stock and has to pay $\pm(S_T - K)$ only when exercised against. In place of (2.5), the terminal conditions on $(L, M)$ now become

\[
dL_T = (-y)_+ \quad \text{and} \quad dM_T = y_+.
\]

Hence, instead of (5.3), the backward induction algorithm (5.2) is initialized here by

\[
u(0, z, x) = (\mu_0 + \mu_1 e^z)\mathbb{1}_{\{x > 0\}} - (\lambda_0 + \lambda_1 e^z)\mathbb{1}_{\{x < 0\}}.
\]
Once the optimal sell and buy boundaries have been obtained, we can compute upper and lower option values using the method developed in Section 3 with the following modifications. Since the terminal cost is \( \{\pm (S_T - K)\}_+ \), the terminal conditions (3.4) (resp. (3.10)) for the option writer (resp. buyer) are replaced by
\[
y^*_T = 0 \quad \text{and} \quad B_T = \{\pm (S_T - K)\}_+ \quad (\text{resp. } -\{\pm (S_T - K)\}_+)
\]
to yield an upper (resp. lower) option value analogous to (3.5) (resp. (3.9)), in which the second term in the right-hand side is replaced by \( e^{-r(T-t)}E[\{\pm (S_T - K)\}_+ | S_t = S, y_{t-} = y] \).

6. CONCLUSION

In the presence of transaction costs, there is a tradeoff between minimizing the risk associated with writing an option (since the writer cannot hedge away the risk entirely by trading in the underlying stock) and keeping the hedging cost at a minimum. It is then natural to use a stochastic control formulation for the problem of hedging options under transaction costs, since the hedger needs to know when to adjust his holding of shares of stock, and by how much. In our approach via risk minimization (penalized by excessive trading), the stochastic control problem is singular and can eventually be reduced to an optimal stopping problem which can be solved, for each \( y \), by backward induction on an approximating Bernoulli walk.

Another consequence of transaction costs is that they create bounds around the theoretical price within which the market price must fall in order not to give rise to an arbitrage opportunity large enough to cover the cost of exploiting it. In our formulation, the upper and lower prices are defined to be the expected total hedging cost based on the respective optimal strategy of the option writer and buyer, and we have developed efficient algorithms to compute them. Our numerical study shows that rebalancing only when the actual hedge ratio gets too far away from the theoretical value is an effective way to reduce the amount of trading of the underlying stock and that this approach is more favorable than rebalancing only on a fixed number of days.
APPENDIX

The reflected follower problem introduced in Section 2.1 has the form

\[ V(t, x) = \inf_{\xi} E \left\{ \int_t^T f(u, X_u) \, du + \int_{[t, T]} h(u) \, d|\xi_u| + g(X_T) \bigg| X_t = x \right\}, \]

where \( X_t = Z_t + \xi_t + K_t \) is the state process, \( Z_t \) is a Brownian motion, and \( \xi_t = \xi_t^{(1)} - \xi_t^{(2)} \) is a left-continuous process of bounded variation with \( \xi_0 = 0 \), expressible as the difference of two nondecreasing processes \( \xi_t^{(i)} \) and with total variation \(|\xi_t| = \xi_t^{(1)} + \xi_t^{(2)}\). Given \( \{Z_t, \ t \geq 0\} \) and \( \{\xi_t, \ t \geq 0\} \), \( K_t \) is a nondecreasing, left-continuous process with \( K_0 = 0 \) to keep \( X_t \geq 0 \) \((0 \leq t \leq T)\). The cost functions for this problem are the following:

(i) a nonnegative, continuous function \( h(t) \) on \([0, T]\), representing a running cost of controlling action per unit time;

(ii) a real-valued, continuous and continuously differentiable function \( g(x) \) on \( \mathbb{R}_+ \), such that \( g'(x) \) is nondecreasing and \( g'(0) = 0 \), representing a terminal cost on the state; and

(iii) a real-valued, continuous function \( f(t, x) \) on \([0, T] \times \mathbb{R}_+\), with continuous gradient \( \partial f(t, x) / \partial x \) which is nondecreasing in \( x \) and satisfies \( \partial f(t, 0) / \partial x \geq 0 \), representing a running cost per unit time on the state.

The functions \( \partial f(t, x) / \partial x \) and \( g'(x) \) are assumed to satisfy a polynomial growth condition in \( x \): for some \( m \geq 1 \) and \( C > 0 \),

\[ 0 \leq \frac{\partial f(t, x)}{\partial x} + g'(x) \leq C(1 + x^m), \quad \text{on} \ [0, T] \times \mathbb{R}_+. \]

Under the assumption that the control problem admits an optimal solution, Karatzas and Shreve (1985) established by probabilistic arguments the connection between singular stochastic control and optimal stopping: There exists an equivalent optimal stopping problem (with absorption at the origin) whose value function is \( U(t, x) = \partial V(t, x) / \partial x \) and whose optimal continuation region corresponds to the no-action region of the reflected follower problem. Specifically,

\[ U(t, x) = \inf_{\tau \in T(t, T)} E \left\{ \int_t^{\tau \wedge \tau_0} \frac{\partial f(u, Z_u)}{\partial x} \, du + h(\tau) 1_{\{\tau < T \wedge \tau_0\}} + g'(Z_T) 1_{\{\tau = T \wedge \tau_0\}} \bigg| Z_t = x \right\}, \]

where \( \tau_0 = \inf\{s \geq t : Z_s = 0\} \). Regarding \( \partial f / \partial x \) as a cost of continuation, \( h \) as a fee for premature termination (i.e., stopping before hitting the origin or running out of time), and \( g' \) as a terminal cost, \( U \) is
the value function of an optimal stopping problem. Moreover, the function \( U(t, \cdot) \) is nondecreasing on \( \mathbb{R}_+ \) with \( U(t, 0) = 0 \).

The bounded variation follower problem, whose connection to optimal stopping was established earlier by Karatzas (1983) using analytic arguments, is a special case of the reflected follower problem, with \( K_t \equiv 0 \) and extending \( f(t, \cdot) \) and \( g(\cdot) \) evenly on the whole of \( \mathbb{R} \). Let \( V_0(t, x) \) be the value function of the bounded variation follower problem (to distinguish it from \( V(t, x) \) of the reflected follower problem). It is obvious that \( V_0(t, \cdot) \) is an even convex function on \( \mathbb{R}_+ \). The spaceship control problem of Bather and Chernoff (1967) cited in the second paragraph of Section 2.1 is a bounded variation follower problem with \( f(t, x) = 0 \), \( h(t) = 1/(T-t) \), and \( g(x) = x^2/2 \). Note that (2.16) requires a slight generalization of the bounded variation follower problem since problem (2.16) is not symmetric about \( x = D(s, z) \) and the control \( x = L - M \) is not applied directly to \( \{Z_s\} \). In order to make effective comparisons of expected costs at nearby points in the control problem (2.16) and thereby to compute \( \partial v(s, z, x)/\partial x \), we make use of the principle of tracking an optimal or nearly optimal path up to a certain stopping time and then jumping on the path. Denoting the four derivatives of \( v(s, z, \cdot) \) at \( x \) by

\[
\Delta^\pm v(s, z, x) = \limsup_{\delta \to 0^\pm} \frac{v(s, z, x + \delta) - v(s, z, x)}{\delta},
\]

\[
\Delta^\pm v(s, z, x) = \liminf_{\delta \to 0^\pm} \frac{v(s, z, x + \delta) - v(s, z, x)}{\delta},
\]

it can be shown by considering the stopping time

(A.1)

\[
\tau^* = \inf\{u \in [s, 0] : L_u + M_u > 0\},
\]

with the convention \( \inf \emptyset = 0 \), that for \( x > D(s, z) \),

\[
\Delta^+_v(s, z, x) \leq \Delta^+_v(s, z, x) \leq u(s, z, x)
\]

\[
\leq E\left\{ \int_s^{\tau^*} g(u, Z_u, x_u) du + G(\tau^*, Z_{\tau^*}, x_{\tau^*}) \bigg| Z_s = z, x_s = x \right\}
\]

\[
\leq \Delta^- v(s, z, x) \leq \Delta^- v(s, z, x).
\]

Making use of the convexity of \( v(s, z, \cdot) \), it follows that \( \Delta^- v(s, z, x) \leq \Delta^+_v(s, z, x) \) for \( x > D(s, z) \) and consequently that \( \partial v(s, z, x)/\partial x \) exists and equals \( u(s, z, x) \) for \( x > D(s, z) \), and that \( \tau^* \) defined in (A.1) with optimal \( (L, M) \) is an optimal stopping time. For \( x < D(s, z) \), we consider \( -v(s, z, x) \) and repeat the argument.
REFERENCES


Table 3.1: Upper and lower option values (3.5) and (3.9) for $S = 100$, $\alpha = r = 0.08$, $\sigma = 0.4$, $T - t = 1.25$, $y = 0$, and transaction cost rates $\lambda = \mu$. Numbers in parentheses indicate standard errors of simulated estimates.

<table>
<thead>
<tr>
<th>K</th>
<th>$\lambda = 0.0$</th>
<th>$\lambda = 0.0005$</th>
<th>$\lambda = 0.005$</th>
<th>$\lambda = 0.05$</th>
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<td>$q = 0$</td>
<td>$q = \alpha$</td>
<td>$q = 0$</td>
<td>$q = \alpha$</td>
</tr>
<tr>
<td>80</td>
<td>32.7706 25.1766</td>
<td>32.8573 (.0127)</td>
<td>33.4226 (.0263)</td>
<td>25.7560 (.0275)</td>
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<tr>
<td>90</td>
<td>26.9621 20.1463</td>
<td>27.0598 (.0152)</td>
<td>27.6412 (.0314)</td>
<td>20.7562 (.0317)</td>
</tr>
<tr>
<td>100</td>
<td>22.0221 16.0099</td>
<td>22.1312 (.0172)</td>
<td>22.6987 (.0360)</td>
<td>16.5720 (.0345)</td>
</tr>
<tr>
<td>120</td>
<td>14.4689 9.9773</td>
<td>14.5494 (.0193)</td>
<td>15.0871 (.0417)</td>
<td>10.4549 (.0388)</td>
</tr>
<tr>
<td></td>
<td>14.3656 (.0192)</td>
<td>13.8865 (.0402)</td>
<td>9.4611 (.0368)</td>
<td>10.0603 (.0780)</td>
</tr>
</tbody>
</table>
FIGURE 2.1: Optimal sell (top) and buy (bottom) boundaries for $\beta = \rho = 0.5$, $\theta = 0$, and $\lambda = \mu = 0.005$. Sell region is above the sell boundary $X_s(s, z)$, buy region is below the buy boundary $X_b(s, z)$, and the no-transaction region $\mathcal{N}$ is between $X_s$ and $X_b$. 
Figure 2.2: Cross-sectional plots of optimal sell (upper) and buy (lower) boundaries. Here, $X_s(s, z)$ and $X_b(s, z)$ are respectively plotted as functions of $z$ with $s = -0.05, -0.10, -0.20, -0.30$, for $\beta = \rho = 0.5, \theta = 0$, and $\lambda = \mu = 0$ (——), $0.0005$ (---), $0.005$ (···), $0.05$ (---). Note that $X_s(s, z) = X_b(s, z) = D(s, z)$ when $\lambda = \mu = 0$. 
Three typical simulated stock price paths ("down", "flat", "up") are shown in the top panels for $S = 100$, $\alpha = r = 0.08$, $q = 0$, $\sigma = 0.4$, and $T - t = 1.25$. The next five rows show the optimal holding $y^*_t$ of stock over time (actual time of trade and number of shares held after each trade) for $\lambda = \mu = 0.005$ and $K = 80, 90, 100, 110, 120$, respectively.
\[ \lambda = 0.0005 \]
\[ K = 80 \]

\[ \lambda = 0.005 \]
\[ K = 80 \]

\[ \lambda = 0.05 \]
\[ K = 80 \]

\[ K = 100 \]

\[ K = 100 \]

\[ K = 100 \]

\[ K = 120 \]

\[ K = 120 \]

\[ K = 120 \]

**Figure 4.2:** Probability distribution of transaction frequency for \( S = 100, K = 80, 100, 120, \alpha = r = 0.08, q = 0, \sigma = 0.4, T - t = 1.25, \) and \( \lambda = \mu. \)
Figure 4.3: Probability distribution of average trade size for $S = 100$, $K = 80, 100, 120$, $\alpha = r = 0.08$, $q = 0$, $\sigma = 0.4$, $T - t = 1.25$, and $\lambda = \mu$. Note that the axes for different values of $\lambda$ are drawn on different scales.
Figure 4.4: Comparison of mean-variance performance of hedging strategies for a short call option (top) and a long call option (bottom). Results for the Black-Scholes strategy (discrete-time replication using unadjusted delta), the Leland strategy (discrete-time replication using adjusted delta), and the Clewlow-Hodges strategy (optimal under utility maximization) are due to Clewlow and Hodges (1997). The new hedging strategy developed in Section 2 (thin solid line) is optimal under risk minimization (penalized by excessive trading). Here, $S = K = 100$, $\alpha = r = q = 0$, $\sigma = 0.3$, $T - t = 1$, and $\lambda = \mu = 0.01$. 