Pricing and hedging Asian options: a recursive integration approach

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An Asian option is a security whose payoff depends on the price average of the underlying asset. It remains a challenge to be able to price an Asian option quickly and accurately, since no explicit pricing formula exists even for most European-style Asian options. In this paper, we demonstrate the method of recursive numerical integration for pricing European-style Asian options, for which early exercise is not allowed. The price average is written in such a way as to allow the evaluation of its density through a recursive sequence of one-dimensional integrals involving the univariate normal distribution. We highlight features of this implementation that are superior to competing methods. However, as recursive numerical integration can prove to be computationally intensive with a large number of fixings, we suggest two alternatives to circumvent this difficulty. The first is based on the relationship between the number of fixings and the option price; the other uses a parametric approximation of the price average density. As a by-product of the first alternative, we are able to deduce the price of a continuously monitored Asian option from similar discretely monitored ones.
1. INTRODUCTION

An option is a security which gives its holder the right to receive a contingent payoff within a specified period of time. For an Asian option, this payoff depends on the price average of an underlying asset. We can broadly classify Asian options according to four features. (1) Provision for early exercise. An Asian option is termed European-style if early exercise of the option is not allowed; otherwise it is said to be American-style. (2) Fixed or floating strike. An Asian option whose payoff is calculated from the difference between the price average and a predetermined strike price belongs to the fixed strike variety. Such an option is sometimes called an average rate option. An Asian option of the floating strike variety has a payoff determined by the difference between the spot asset price and the price average. Such an option is sometimes called an average strike option. (3) Arithmetic or geometric averaging. The price average can be computed from the set of underlying asset prices through arithmetic averaging or geometric averaging. (4) Discrete or continuous fixing. The underlying asset prices that constitute the price average can be collected on a discrete set of dates or over a continuous interval of time.

Options utilizing the geometric average can be priced quite explicitly, since the geometric average of lognormal variates is itself lognormally distributed. Black-Scholes-type pricing formulas are easy to obtain; see, for example, Turnbull and Wakeman (1991). We will therefore focus mainly on Asian options defined with the arithmetic average. We will also restrict our discussion to European-style Asian options.

Previous work on numerical analysis of European-style Asian options can be placed into four broad categories. Kemna and Vorst (1990) demonstrated how Monte Carlo simulation can be used to price Asian options and implemented a variance reduction technique to improve the precision of price estimates. Efforts involving density approximation include Carverhill and Clewlow (1990), who used Fourier transform to calculate the price average density which they then used directly in option pricing, and Turnbull and Wakeman (1991) and Levy (1992), who relied on a lognormal approximation of the true distribution to obtain Black-Scholes-type pricing formulas. Analysis by binomial tree can be attributed to Barraquand and Pudet (1996), who improved on the proposed tree-based implementation of Hull and White (1993). They also showed that the diffusion partial differential equation (PDE) for the Asian option value is convection-dominated. This means that standard PDE methods converge extremely slowly, are prone to oscillatory solutions, and may produce inaccurate solutions because they introduce spurious diffusion into the problem. As a result, approaches based on finite-difference techniques...
have often relied on a change of numeraire (or similarity reduction in PDE terminology) to reduce the option pricing problem to a one-dimensional PDE in time and a Markovian state variable; see Rogers and Shi (1995) and Alziary, Décamps, and Koehl (1997). The PDE literature then differs mostly in the techniques used to solve the resulting one-dimensional PDE; for example, Zvan, Forsyth, and Vetzal (1997) used a high-order nonlinear flux limiter for the convection term in the PDE. Dewynne and Wilmott (1995) and Andreasen (1998) treated the case of discretely sampled Asian options. Using a combination of methods, Little and Pant (2000) numerically solved PDEs for the moments of the arithmetic average and used the Johnson family of curves to approximate the actual distribution of the average for pricing purposes.

Apart from numerical approaches to the problem, there are quasi-analytic methods. Geman and Yor (1993) derived an analytic expression for the Laplace transform in maturity for continuous Asian call options and numerical inversion of this transform was considered briefly in Geman and Eydeland (1995) and more extensively in Fu, Madan, and Wang (1998); see Craddock, Heath, and Platen (2000) for a survey of the various numerical inversion techniques that are applicable to this problem. Rogers and Shi (1995) applied a conditioning technique to obtain a lower bound for option values as a two-dimensional integral to be evaluated numerically. Inspired by this approach, Chalasani, Jha, and Varikooty (1998) devised an algorithm based on a $n$-period binomial tree model to compute this lower bound more readily. We will comment on these quasi-analytic approaches in the context of continuous fixings in Section 6.

The rest of the paper is organized as follows. In Section 2 we lay out the setting for Asian options with discrete fixings and highlight some results associated with geometric averaging, particularly the convergence of option values with increased frequency of fixings. In Section 3 we derive a sequence of recursive formulas that gives the density of the arithmetic price average. We propose in Section 4 the method of recursive numerical integration for the evaluation of the density of this price average. We price a variety of fixed strike Asian options using the densities obtained in this way and show that these price estimates are accurate against Monte Carlo simulation. Details for dealing with floating strike options are provided in Section 5. Next, two alternatives are presented that will improve the timeliness of our algorithm for online pricing of options. In Section 6 we examine how the frequency of fixings affects the price of an option; in Section 7 we develop a mixed density approximation to the true density of the price average and show that it offers a substantial improvement over the more commonly used lognormal approximation. We conclude with comments in Section 8.
2. OPTIONS WITH DISCRETE FIXINGS

In the standard Black-Scholes environment, the asset price process is governed by the usual stochastic differential equation (SDE):

\[ dS_t = rS_t \, dt + vS_t \, dB_t, \quad S_0 > 0, \]

where \( r \) is the short-term interest rate, \( v \) is the instantaneous standard deviation of the asset return and \( \{B_t\} \) is a standard Brownian motion under the risk-neutral measure \( Q \). Let \( \tilde{r} = r - v^2/2 \). A solution of the SDE is given by

\[ S_t = S_0 \exp(\tilde{r}t + vB_t), \quad S_0 > 0. \] (1)

In the case of discrete fixings, we assume that there are \( m + 1 \) equally spaced monitoring dates between 0 and the expiration date \( T \), and let \( \Delta = T/m \). The asset prices that constitute the price average are given by \( S_j := S_{j\Delta}, \ j = 0, 1, \ldots, m \). In the case of arithmetic (resp. geometric) averaging, the path-dependent quantity is \( A_k \) (resp. \( G_k \)), given by

\[
A_k = \frac{1}{k+1} \sum_{j=0}^{k} S_j \quad \text{and} \quad G_k = \left( \prod_{j=0}^{k} S_j \right)^{1/(k+1)}.
\]

We can now write down the pricing formulas for Asian options with discrete fixings. The initial values (i.e., at date 0) of European-style Asian options with fixed strike \( K \) are

\[
C_0 = e^{-rT} E[A_m - K]^+ \quad \text{for the call and} \quad P_0 = e^{-rT} E[K - A_m]^+ \quad \text{for the put;} \quad (2)
\]

those of floating strike Asian options are

\[
\check{C}_0 = e^{-rT} E[S_m - A_m]^+ \quad \text{for the call and} \quad \check{P}_0 = e^{-rT} E[A_m - S_m]^+ \quad \text{for the put.} \quad (3)
\]

In each case, we replace \( A_m \) by \( G_m \) if geometric averaging is used instead of arithmetic averaging. Here, the expectations are taken under the risk-neutral measure \( Q \).

2.1 Geometric Averaging

When the geometric average is used, Black-Scholes-type pricing formulas are easy to obtain even with discrete fixings. We have the following results, where \( N(\cdot) \) denotes the standard normal distribution function. We defer the pricing of arithmetic average options to Sections 3–5.
Lemma 1. Under $Q$, $(W_1, W_2) := (\log S_m, \log G_m)$ has a bivariate normal distribution with mean $(\mu_1, \mu_2)$, variance $(\sigma_1^2, \sigma_2^2)$ and correlation coefficient $\rho$, where

$$
\mu_1 = \bar{r}T + \log S_0, \quad \mu_2 = \mu_1 - \frac{\bar{r}T}{2}, \quad \sigma_1 = \sqrt{\frac{2m+1}{6(m+1)}}, \quad \sigma_2 = \sigma_1 \sqrt{\frac{2m+1}{6(m+1)}} \quad \text{and} \quad \rho = \frac{\sigma_1}{2\sigma_2}.
$$

Lemma 2. With geometric averaging, the initial value of a fixed strike Asian call option is

$$
c_0 = e^{-rT}E[GM-K]^+ = S_0e^{\tilde{\sigma}_m^2/2-\kappa}N(\tilde{\gamma}_m + \tilde{\sigma}_m) - KN(\tilde{\gamma}_m), \quad (4)
$$

and that of a floating strike Asian call option is

$$
\tilde{c}_0 = e^{-rT}E[S_m - G_m]^+ = S_0 \left[ N \left( \frac{\kappa}{\tilde{\sigma}_m} \right) - e^{\tilde{\sigma}_m^2/2-\kappa}N \left( \frac{\kappa}{\tilde{\sigma}_m} - \tilde{\sigma}_m \right) \right], \quad (5)
$$

where

$$
\tilde{\sigma}_m = \sqrt{\frac{T(2m+1)}{6(m+1)}}, \quad \tilde{\gamma}_m = \frac{\log(S_0/K) + \bar{r}T/2}{\tilde{\sigma}_m}, \quad \text{and} \quad \kappa = \left( r + \frac{v^2}{2} \right) \frac{T}{2}.
$$

2.2 From Discrete to Continuous Fixings

As we allow the number of fixings to increase to infinity, i.e., $m \to \infty$, the path-dependent quantities converge to their continuous analogues:

$$
A_m \to A_T := \frac{1}{T} \int_0^T S_u \, du \quad \text{and} \quad G_m \to G_T := \exp \left( \frac{1}{T} \int_0^T \log S_u \, du \right).
$$

The rates at which the discrete Asian option prices converge to the continuous prices would be of interest. In particular, we note that $c_0 \to c_0^* := e^{-rT}E[G_T-K]^+$ and $\tilde{c}_0 \to \tilde{c}_0^* := e^{-rT}E[S_T-G_T]^+$, where $c_0^*$ (resp. $\tilde{c}_0^*$) is the value of a fixed (resp. floating) strike continuous geometric average option given by (4) (resp. (5)) with $\tilde{\sigma}_m$ replaced by its limiting value $\tilde{\sigma}^* = \sqrt{\frac{T}{3}}$.

We note the following convergence result for fixed strike discrete geometric average options.

We will discuss the convergence of arithmetic average option values in Section 6.

Theorem 1. The value of a fixed strike discrete geometric average option converges to the value of a corresponding continuous option at the rate of $O(m^{-1})$. Specifically,

$$
m(c_0^* - c_0) \sim S_0 e^{-(r+v^2/6)T/2} \left[ \tilde{\sigma}^* N(\gamma^* + \tilde{\sigma}^*) - \frac{\gamma^* - \tilde{\sigma}^*}{4} n(\gamma^* + \tilde{\sigma}^*) \right] + \frac{Ke^{-rT} \gamma^* n(\gamma^*)}{4},
$$

where $\tilde{\sigma}^* = \sqrt{\frac{T}{3}}$ and $\gamma^* = (\log(S_0/K) + \bar{r}T/2)/\tilde{\sigma}^*$. 

5
Proof of Theorem 1. We write out the difference \( c_0^* - c_0 \) and use the following Taylor series approximations for sufficiently large \( m \):

\[
\exp \left[ -\frac{v^2 T}{6(2m + 2)} \right] = 1 - \frac{v^2 T}{6(2m + 2)} = 1 - \frac{\sigma^*}{2(2m + 2)},
\]

\[
N \left[ \gamma^* + \frac{\gamma^*}{2(2m + 1)} \right] \approx N(\gamma^*) + \frac{\gamma^*}{2(2m + 1)}.
\]

\[
N \left[ \frac{\gamma^*}{2(2m + 1)} + \frac{\sigma^*}{2(2m + 1)} \right] \approx N(\gamma^*) + \frac{\gamma^*}{2(2m + 1)}.
\]

3. FIXED STRIKE OPTIONS

We turn our focus now to Asian options defined with the arithmetic average. We will deal with fixed strike options in this section and floating strike options in Section 5. Using (1), the asset prices that constitute the arithmetic average can be written as

\[
S_j = S_0 \exp(U_j), \quad j = 0, 1, \ldots, m,
\]

with \( U_0 = 0 \) and \( U_j = X_1 + X_2 + \cdots + X_j \), where the \( X_i \)'s are independent \( N(\mu, \sigma^2) \) random variables such that \( \mu = \bar{r} \Delta \) and \( \sigma = v\sqrt{\Delta} \). Then the arithmetic average \( A_m \) at date \( m \) can be expressed as

\[
A_m = S_0 (1 + e^{U_1} + \cdots + e^{U_m})/(m + 1).
\]

At date \( m - i \), consider the sum of the remaining fixings: \( e^{U_{m-i}} + e^{U_{m-i+1}} + \cdots + e^{U_m} \), which we will write in the form \( e^{U_{m-i}} \times (i + 1)Y_i \), with \( Y_0 = 1 \). Then

\[
Y_i = \frac{1}{i + 1} + \sum_{j=i+1}^{m} e^{X_{m-i+1} + \cdots + X_{m-j}} = 1 + \frac{Y_{i-1}e^X}{i + 1}, \quad i = 1, \ldots, m, \tag{6}
\]

where \( X \) is a \( N(\mu, \sigma^2) \) random variable independent of \( Y_{i-1} \). It is useful to think of \( Y_i \) as essentially the arithmetic average of the final \( i + 1 \) fixings. We have \( A_m = S_0 Y_m \).

Let \( f_i \) be the density of \( Y_i \) \((i = 1, \ldots, m)\). With \( n(\cdot) \) denoting the standard normal density function and \( h_i = 1/(i + 1) \), we obtain

\[
f_1(x) = \frac{1}{\sigma(x - h_1)} n \left( \frac{\log(x - h_1) - \log h_1 - \mu}{\sigma} \right), \quad x > h_1, \tag{7a}
\]

and, using (6) for \( i = 2, \ldots, m \),

\[
f_i(x) = \frac{1}{\sigma(x - h_i)} \int_{h_{i-1}}^{\infty} n \left( \frac{\log(x - h_i) - \log(yh_i/h_{i-1}) - \mu}{\sigma} \right) f_{i-1}(y) \, dy, \quad x > h_i. \tag{7b}
\]
Through the recursive definitions (7a)–(7b), we can obtain \( f_m \) and then compute the value of an arithmetic average call option from (2) as follows:

\[
C_0 = e^{-rT} \int_{h_m}^\infty (S_0x - K)^+ f_m(x) \, dx = e^{-rT} \int_{K/S_0}^\infty (S_0x - K) f_m(x) \, dx \quad \text{if } K \geq h_m S_0, \tag{8a}
\]

\[
C_0 = h_m S_0 \left( \frac{e^{r\Delta} - e^{-rT}}{e^{r\Delta} - 1} \right) - e^{-rT} K \quad \text{if } K < h_m S_0. \tag{8b}
\]

Note that in the latter case, the initial price of the option is independent of the volatility \( \nu \) of the underlying asset. For the put option,

\[
P_0 = e^{-rT} \int_{h_m}^{K/S_0} (K - S_0x) f_m(x) \, dx \quad \text{if } K > h_m S_0 \quad \text{and} \quad P_0 = 0 \quad \text{if } K \leq h_m S_0. \tag{9}
\]

In practice, the arbitrage-free put option value should be obtained using the put-call parity relation:

\[
P_0 = C_0 + K e^{-rT} - h_m S_0 \left( \frac{e^{r\Delta} - e^{-rT}}{e^{r\Delta} - 1} \right). \tag{10}
\]

4. RECURSIVE NUMERICAL INTEGRATION

The densities (7a)–(7b) have unbounded support. To implement a numerical scheme for computing these densities, we need to truncate the densities. For this purpose, we make two considerations. First, the value of each density outside its “effective range” should be negligible. Second, each truncated density should retain as much as possible of its coverage probability (i.e., the integral of a truncated density over its effective range should be as close to 1 as possible).

When these requirements are imposed on the densities (7a)–(7b), we arrive at the following results regarding the appropriate choices of “truncation points;” see Appendix for details.

**Proposition 1.** Assume that \( 0 < \alpha < 1 \) is given.

(a) Let \( \varepsilon_1^*(\alpha) = (\sigma h_1 \sqrt{2\pi})^{-1} \exp((\sigma^2 - 2\mu - B_1^*(\alpha)^2)/2) \), \( \bar{\varepsilon}_1^*(\alpha) = h_1 + h_1 \exp(-\sigma B_1^*(\alpha) - (\sigma^2 - \mu)) \) and \( \bar{\pi}_1^*(\alpha) = h_1 + h_1 \exp(\sigma B_1^*(\alpha) - (\sigma^2 - \mu)) \), with \( B_1^*(\alpha) = \sigma + N^{-1}((1 + \alpha)/2) \). Then

\[
f_1(x) \leq \varepsilon_1^*(\alpha) \quad \text{whenever } x \notin (\bar{\varepsilon}_1^*(\alpha), \bar{\pi}_1^*(\alpha)) \quad \text{and} \quad \int_{\bar{\varepsilon}_1^*(\alpha)}^{\bar{\pi}_1^*(\alpha)} f_1(x) \, dx \geq \alpha.
\]

(b) For \( i \geq 2 \), let \( \varepsilon_i^*(\alpha) = h_{i-1}(\sigma h_i \bar{\varepsilon}_{i-1}^*(\alpha) \sqrt{2\pi})^{-1} \exp((\sigma^2 - 2\mu - B_i^*(\alpha)^2)/2) \), \( \bar{\varepsilon}_i^*(\alpha) = h_i + h_i \bar{\varepsilon}_{i-1}^*(\alpha) h_{i-1}^{-1} \exp(-\sigma B_i^*(\alpha) - (\sigma^2 - \mu)) \) and \( \bar{\pi}_i^*(\alpha) = h_i + h_i \bar{\varepsilon}_{i-1}^*(\alpha) h_{i-1}^{-1} \exp(\sigma B_i^*(\alpha) - (\sigma^2 - \mu)) \).
take approximation proposed by Levy (1992). Throughout this section and the rest of the paper, we
of Barraquand and Pudet (1996). We will also comment on the adequacy of the lognormal
of recursive numerical integration (RNI) against the forward shooting grid (FSG) algorithm
estimates obtained using Monte Carlo (MC) simulation. We will compare the performance

First we note from (8b) that exact values can be obtained for all Asian options with parameters

With the truncation points $x_i^*(\alpha)$ and $\pi_i^*(\alpha)$ defined for $i = 1, \ldots, m$ as in Proposition 1, we compute the densities (7a)–(7b) recursively as follows. For $\delta > 0$ (grid size) and $\bar{\alpha} > 0$ (coverage probability of $f_{mn}$), let $\alpha = \bar{\alpha}^j/m$, $\bar{k}_i = [x_i^*(\alpha)/\delta]$ and $\bar{\bar{k}}_i = [\pi_i^*(\alpha)/\delta]$. When $\delta$ is sufficiently small, we can treat $f_i(y)$ as 0 for $y \leq \bar{k}_i\delta$ and $y \geq \bar{\bar{k}}_i\delta$, and approximate $f_i(y)$ by $f_i(j\delta)$ for $(j - 1/2)\delta \leq y < (j + 1/2)\delta$. Then we obtain $f_{i+1}(x)$ via the sum $\delta \sum_{j=\bar{\bar{k}}_{i+1}}^{\bar{k}_{i+1}} \psi_{i+1}(x, y_j) f_i(y_j)$, where $y_j = j\delta$ and

Note that in this recursive algorithm, we need only compute $f_{i+1}(x)$ for $x = j\delta$, $j = \bar{k}_{i+1} + 1, \ldots, \bar{\bar{k}}_{i+1} - 1$. Moreover, after the $i$th iteration we only have to store the $\bar{k}_i - \bar{\bar{k}}_i - 1$ values of $y_j$ and of $f_i(y_j)$ to be used in the next iteration. We estimate $C_0$ with the sum $\delta e^{-rT} \sum_{j=\bar{\bar{k}}_{m+1}}^{\bar{k}_{m+1}} (S_0x_j - K)^+ f_m(x_j)$, where $x_j = j\delta$.

Our method is applied to a variety of Asian call options. We take as benchmark the price estimates obtained using Monte Carlo (MC) simulation. We will compare the performance of recursive numerical integration (RNI) against the forward shooting grid (FSG) algorithm of Barraquand and Pudet (1996). We will also comment on the adequacy of the lognormal approximation proposed by Levy (1992). Throughout this section and the rest of the paper, we take $S_0 = 100$ and use the following sets of parameters:

Set A: $r = 0.09, \ T = 1, \ m = 52$; \hspace{1cm} (12a)

Set B: $r = 0.10, \ m = 91 \ (T = 91/365, 182/365)$ or 121 ($T = 364/365$). \hspace{1cm} (12b)

The strike price $K$ and volatility $v$ will be allowed to vary in both sets of parameters.

4.1 Validation

First we note from (8b) that exact values can be obtained for all Asian options with parameters (12a) and $K < 100/53$. In particular, the exact price of zero-strike options is 95.63 for all volatilities $v$. Correspondingly, evaluating the integral in (8a) using the RNI density and with $h_m = 1/53$ as the lower integration limit yields $C_0 = 95.63$ for zero-strike options at several volatilities: $v = 0.05, 0.10, 0.30, 0.50$. The maximum relative error in these four price estimates is 0.0024%, which shows that RNI can potentially produce remarkably accurate price estimates.
4.2 Monte Carlo Simulation

We price Asian options with parameters (12a) and \( K = 90, 95, 100, 105, 110 \) for four volatilities: \( v = 0.05, 0.10, 0.30, 0.50 \). The price estimates are reported in Table 1 under the columns headed RNI. Corresponding MC estimates, as reported in Levy and Turnbull (1992), are given in the columns headed MC. The two sets of estimates are remarkably close in value.

However, MC simulation yields different estimates each time the procedure is repeated. To gauge the variability in price estimates obtained using MC simulation, we perform the following calculations with \( K = 100 \). For each volatility \( v \) we generate 1000 repetitions of a 10000 simulation series from which a MC estimate is calculated; see Kemna and Vorst (1990).\(^3\) The resulting 1000 price estimates for each volatility are summarized in a boxplot (see Figure 1).

MC simulation produces relatively precise price estimates in the case of low volatility (e.g., \( v = 0.10 \)). However, the precision of MC estimates deteriorates when \( v \) (or \( T \)) increases, as is evident from the summary statistics given for each of the four simulation runs we examine in Figure 1. Each mean estimate (standard error) is calculated from the 1000 MC estimates obtained by simulation. It is clear that the standard error of MC estimates increases when \( v \) increases, indicative of the loss of precision associated with higher volatility. We also report, for each volatility \( v \), the range of MC estimates we obtained and, assuming the mean estimate is the true value of the option, the percentage of MC estimates that were in error by an amount more than the corresponding RNI price estimate. For example, when \( v = 0.50 \), we would obtain an MC estimate that is inferior to the RNI estimate 95.4% of the time! These measures suggest that RNI is superior to MC simulation when moderate to high volatility is present. The higher volatility requires more sampling repetitions to be run in simulation to obtain an estimate of a given precision. The larger standard error in the estimates would have to be reduced with a longer simulation series, at the expense of computational efficiency.

4.3 Lognormal Approximation

Turnbull and Wakeman (1991) employed an Edgeworth series expansion about an approximating lognormal distribution to obtain an approximate density of the price average. Levy (1992)
specialized this method using the Wilkinson approximation by matching only the first two moments of the average. Specifically, \( \log Y_m \) is assumed to be distributed as \( N(\alpha_m, \beta_m^2) \), with

\[
\alpha_m = 2 \log E(Y_m) - \log E(Y_m^2)/2 \quad \text{and} \quad \beta_m = \sqrt{\log E(Y_m^2) - 2 \log E(Y_m)},
\]

where

\[
E(Y_m) = h_m \left( \frac{e^{(m+1)r \Delta} - 1}{e^{r \Delta} - 1} \right),
\]

\[
E(Y_m^2) = h_m^2 \left[ \frac{2e^{r \Delta} (e^{mr \Delta} - 1)}{e^{r \Delta} - 1} + \frac{e^{(m+1)(2r+v^2) \Delta} - 1}{e^{(2r+v^2) \Delta} - 1} + 2e^{2(2r+v^2) \Delta} (e^{(m-1)(2r+v^2) \Delta} - 1) \right] - \frac{2e^{(3r+v^2) \Delta} (e^{(m-1)r \Delta} - 1)}{(e^{r \Delta} - 1)(e^{r \Delta})}.
\]

As discussed in Levy (1992), the lognormal approximation works well in the case of low volatility, but the adequacy quickly degenerates as volatility increases, as illustrated in Figure 2. The quantile-plots of “standardized” \( \log Y_m \), for parameters (12a) and \( v = 0.10, 0.30, 0.50 \), against the standard normal distribution show that for moderate to high volatility, the lognormal approximation does not capture the tails of the true distribution accurately.\(^4\) This is the reason for the mispricing of both in-the-money and out-of-the-money Asian call options when the lognormal approximation is used. We also note that there is no substantial improvement in the accuracy of price estimates when the Edgeworth series is used. For moderate to high volatility, the Edgeworth series approximation tends to underprice in-the-money options and overprice out-of-the-money options; see results of Levy and Turnbull (1992).

\[\text{INSERT FIGURE 2 ABOUT HERE}\]

### 4.4 Forward Shooting Grid Algorithm

While Hull and White (1993) demonstrated the potential of tree methods in the pricing of Asian options, the complexity of the trees renders the methods difficult to implement and computationally expensive. Barraquand and Pudet (1996) attempted to overcome this problem with the FSG method, a modification of the Cox-Ross-Rubinstein (CRR) binomial approximation, by first “growing” the binomial tree forward and then moving backwards through the branches to price an Asian option (via the Feymann-Kac characterization of the PDE solution).

In Table 2 we compare RNI against the FSG algorithm, using the parameter values (12b) and \( K = 95, 100, 105 \) for three volatilities: \( v = 0.10, 0.30, 0.50 \). MC estimates are obtained as before (see footnote 3) and serve as the basis for comparison of the two methods. We find that
the performance of the FSG algorithm is unsatisfactory for large \(v\) and/or large \(T\). In such cases it is likely that the FSG algorithm needs to be implemented with finer state quantization, a consequence of the CRR binomial approximation it adopts. Accuracy was perhaps compromised in Barraquand and Pudet (1996) because their binomial walk employed steps that were too large.

4.5 Discussion

RNI is designed to price options with discrete fixings. The key advantage of the method is that the accuracy and precision of price estimates do not deteriorate significantly as \(v\) (or \(T\)) increases, unlike MC simulation which loses precision and lognormal approximation which suffers from inaccuracy. *Once the terminal density \(f_m\) is obtained, option prices can be computed for any \(S_0\) and \(K\).* The computed densities can be stored for future online use, which makes for prompt evaluation of Asian option prices. In addition, the density is useful for the evaluation of hedge parameters. For example, at the start of the averaging period, the delta \(D_0\) and gamma \(G_0\) of a call option are given respectively by direct differentiation of (8a) and (8b):

\[
D_0 = e^{-rT} \int_{K/S_0}^{\infty} x f_m(x) \, dx \quad \text{and} \quad G_0 = e^{-rT}(K^2/S_0^3) f_m(K/S_0) \quad \text{if} \quad K \geq h_m S_0,
\]

\[
D_0 = h_m \left( \frac{e^{r\Delta} - e^{-rT}}{e^{r\Delta} - 1} \right) \quad \text{and} \quad G_0 = 0 \quad \text{if} \quad K < h_m S_0.
\]

For a fixed \(m\) the algorithm for RNI calls for the choice of two parameters: grid size \(\delta\) and “final” coverage probability \(\alpha\). While a simple choice of \(\alpha = 0.9999\) is sufficient for most practical purposes, our choice of \(\delta\) must provide enough grid points to capture accurately the shape of a recursive density \(f_i(x)\) and the convoluting density in (11) (particularly close to the modes) for the computation of the next density \(f_{i+1}(x)\). In our implementation, we find that a good guide for this choice is to have about 50 grid points in the interval \((\underline{x}_i^*(\alpha), \overline{x}_i^*(\alpha))\). As a consequence of the trapezoidal integration scheme we have adopted, we find that our price estimates converge as \(\delta \to 0\).

5. Floating Strike Options

We now shift our focus to European-style floating strike Asian options, whose terminal payoffs are determined by the difference between the terminal spot price of the underlying asset and the average asset price.
5.1 Change of Numeraire

For floating strike Asian options, Hansen and Jørgensen (1997), as well as Andreasen (1998), demonstrated the usefulness of a change of numeraire from the riskless bond to the underlying asset. This is achieved through a change of measure from \( Q \) to \( \tilde{Q} \) with \( d\tilde{Q}/dQ = \xi_T \), where \( \xi_t = e^{-rt}S_t/S_0 \). As a consequence,

\[
dS_t = (r + v^2) dt + vS_t d\tilde{B}_t, \quad \text{i.e.,} \quad S_t = S_0 \exp((r + v^2/2)t + \tilde{B}_t), \quad S_0 > 0,
\]

where \( \{\tilde{B}_t\} \) is a standard Brownian motion under \( \tilde{Q} \) given by \( \tilde{B}_t = B_t - vt \). This leads to the following simplification assuming continuous fixings:

\[
e^{-r(T-t)} E_t [S_T - A_T]^+ = e^{-r(T-t)} \tilde{E}_t \left[ \frac{\xi_t}{\xi_T} (S_T - A_T)^+ \right] = S_t \tilde{E}_t [1 - R_T]^+, \tag{13}
\]

where \( R_t = A_t/S_t \). Here, \( \tilde{E} \) denotes expectation with respect to the measure \( \tilde{Q} \).

Keeping with earlier notation, we will let \( R_k = A_k/S_k \) denote the discretely monitored ratio at date \( k \). It follows from (3) and (13) that the respective discretely monitored option prices at date 0 are given by

\[ \tilde{C}_0 = S_0 \tilde{E}[1 - R_m]^+ \text{ for the call and } \tilde{P}_0 = S_0 \tilde{E}[R_m - 1]^+ \text{ for the put.} \]

With \( X_i, U_j \) and \( Y_k \) defined in Section 3, noting that the \( X_i \)'s have mean \((r + v^2/2)\Delta\) under \( \tilde{Q} \), now define \( \tilde{X}_i, \tilde{U}_j \) and \( \tilde{Y}_k \) by

\[ \tilde{X}_i = -X_i, \quad \tilde{U}_j = \tilde{X}_1 + \cdots + \tilde{X}_j \quad \text{and} \quad \tilde{Y}_k = (1 + e^{\tilde{U}_1} + \cdots + e^{\tilde{U}_k})/(k + 1). \]

It follows that the \( \tilde{X}_i \)'s are independent \( N(\tilde{\mu}, \sigma^2) \) random variables, where \( \tilde{\mu} = -(r + v^2/2)\Delta \). Moreover, since

\[
R_k = \frac{A_k}{S_k} = \frac{Y_k}{e^{\tilde{U}_k}} = \frac{1}{k+1} \left[ 1 + e^{-X_k} + e^{-(X_k+X_{k-1})} + \cdots + e^{-(X_k+\cdots+X_1)} \right],
\]

we conclude that \( R_k \overset{\text{d}}{=} \tilde{Y}_k \). In particular, the densities of \( \{\tilde{Y}_k\} \) can be obtained through the same recursive definitions (6), albeit with the new \( \{\tilde{X}_i\} \) replacing \( \{X_i\} \). Specifically, we can obtain the density \( \tilde{f}_m \) of \( R_m \), as before, by recursive numerical integration, through (7a)–(7b), with \( \tilde{\mu} \) replacing \( \mu \). Therefore, the option values are

\[ \tilde{C}_0 = S_0 \int_{h_m}^1 (1 - x) \tilde{f}_m(x) \, dx \quad \text{and} \quad \tilde{P}_0 = S_0 \int_{1}^{\infty} (x - 1) \tilde{f}_m(x) \, dx. \tag{14} \]
5.2 Results and Discussion

If we denote the fixed strike option prices by \( C_0 := C_0(S_0, K; r, v, m) \) for the call and \( P_0 := P_0(S_0, K; r, v, m) \) for the put, when the initial asset price is \( S_0 \), the strike price is \( K \), the riskless rate is \( r \), the volatility is \( v \) and there are \( m + 1 \) (equally spaced) monitoring dates, then from (8a), (9) and (14), we have

\[
\tilde{C}_0 = S_0 e^{rT} P_0(1, 1; -r, v, m) \quad \text{and} \quad \tilde{P}_0 = S_0 e^{rT} C_0(1, 1; -r, v, m).
\]

We apply RNI to floating strike call options with parameters (12b) and \( v = 0.10, 0.20, 0.40 \). We compare our results against the FSG algorithm and MC simulation in Figure 3. The MC estimates are obtained using a variance reduction technique.\(^5\) We again rely on 1000 repetitions of a 10000 simulation series to gauge the precision of the MC estimates.

**INSERT FIGURE 3 ABOUT HERE**

We observe that RNI estimates are comparable to MC estimates and are significantly more accurate than FSG estimates in most cases. However, unlike MC simulation, the precision of RNI estimates is not compromised as volatility increases. In general, the desirable properties of RNI estimates noted earlier for fixed strike options continue to hold here.

6. OPTIONS WITH CONTINUOUS FIXINGS

While practically all Asian options traded in the market are discretely monitored, it would be of interest to examine how the number of monitoring dates affects the price of an Asian option. In particular, we will demonstrate how the price of continuously monitored Asian options can be approximated by the prices of corresponding ones that are discretely monitored. One should note that the PDE and quasi-analytic methods in the literature are developed in the context of continuous fixings.

6.1 Frequency of Fixings and Convergence of Option Values

We first investigate how the price estimate varies with the number of monitoring dates \( m \). As an illustration, we apply RNI to fixed strike Asian options with parameters \( S_0 = 100, r = 0.09, v = 0.30, T = 1 \), various strike prices \( K \) and numbers of monitoring dates: \( m = 26, 52, 104, 208, 416, 832 \). The results are recorded in Table 3. The estimates indicate strongly that there is a limiting option value when \( m \) gets infinitely large, as suggested by the diminishing increase in the option price whenever \( m \) is doubled.
Indeed, as pointed out in Section 2.2, we can view \( A_T \) as the limit, as \( m \to \infty \), of its discrete counterpart \( A_m = S_0 Y_m \). (Specifically, for each fixed state \( \omega \), \( A_m \to A_T \) as \( m \to \infty \), following the definition of a Riemann integral.) Our numerical results suggest that RNI can be used to approximate the density of \( A_T \) if we choose a large enough \( m \). This fact and Theorem 1 for geometric averaging suggest approximating the option price for large \( m \), holding all other parameters constant, by a function of the form \( \frac{C_0}{\beta m} - 1 \), where \( C_0^* > 0 \) and \( \beta \) are parameters to be determined. In particular, \( C_0^* \) is the price of the Asian option when the average is computed continuously since \( m^{-1} \to 0 \) as \( m \to \infty \). For the Asian options considered in Table 3, their continuous values are reported in the final column using the scheme LS-6 to be described shortly. We note that in these examples, the price of an option with weekly fixings differs by as much as 0.5% from the price of a corresponding option with continuous fixings.

One simple scheme for estimating the constants \( C_0^* \) and \( \beta \) consists of evaluating the option prices using RNI based on \( m \) and \( 2m \) monitoring dates (denote these prices by \( C_1 \) and \( C_2 \), respectively), and then solving the two equations that result: \( C_k = C_0^* - \beta (km)^{-1} \) for \( k = 1, 2 \), to give

\[
C_0^* = 2C_2 - C_1 \quad \text{and} \quad \beta = 2m(C_2 - C_1).
\]

This method is commonly known as Richardson’s extrapolation. Implementing this scheme with \( m = 104 \) and \( m = 416 \) for the Asian options considered in Table 3 yields estimates respectively under the columns headed RE-104 and RE-416 in Table 4.

Another approach is to fit a regression function of the form \( C_0^* - \beta m^{-1} \) to the RNI estimates and obtain \( C_0^* \) and \( -\beta \) respectively as the least squares estimates of the intercept and gradient for the regression line. Results using the regression approach are reported under the columns headed LS-3 and LS-6. For LS-3, only the first three RNI estimates (corresponding to \( m = 26, 52, 104 \)) are used; for LS-6, all six RNI estimates in Table 3 are used to compute the least squares coefficients. It is clear from Table 4 that using RNI values with \( m \leq 104 \) produces estimates of \( C_0^* \) and \( \beta \) that are of comparable accuracy as those estimates obtain using RNI values associated with larger values of \( m \). This observation leads to substantial savings in computational time.

INSERT TABLE 4 ABOUT HERE
Finally, we discuss other approaches in the literature. Geman and Yor (1993) derived a formula for the Laplace transform in maturity of a continuous Asian call option by relating the integral of an exponential of Brownian motion to Bessel processes. However, as Rogers and Shi (1995) pointed out, “numerical inversion of this Laplace transform seems likely to be slow, and no simple analytic inversion has been found to date.” Indeed, Geman and Eydeland (1995) and Fu, Madan, and Wang (1998) implemented numerical schemes for inverting the Laplace transform and found that the algorithms can become unstable for small volatility and/or short maturity, i.e., for small values of \( v^2(T-t) \).

Instead, by exploiting a scaling property of Brownian motion, Rogers and Shi (1995) showed that the pricing problem can be reduced to solving a parabolic PDE in time and one (rather than two) state variable. Moreover, a convenient lower bound (R-LB) of the option price was given. The R-LB is based on conditioning on another random variable \( Z \) and using the fact that \( E(X^+|Z) \geq (E(X|Z))^+ \). Taking \( Z \) to be \( \int_0^T B_u du \), the R-LB takes the form \( E[E(A_T|Z) - K]^+ \), which is a two-dimensional integral that can be evaluated numerically. Note that this choice of \( Z \) is essentially the geometric average. There are two related analyses in the literature. Zvan, Forsyth, and Vetzal (1997) implemented a more sophisticated algorithm to solve the one-dimensional PDE, employing a high-order nonlinear flux limiter for the convection term to improve the accuracy of the solution. To make the R-LB approach amenable to a binomial tree model, Chalasani, Jha, and Varikooty (1998) suggested conditioning on the position-sums (rather than the continuous geometric average) in the tree to compute their lower bound (C-LB).

A comparison of our price estimates with the PDE results of Rogers and Shi (1995) and Zvan, Forsyth, and Vetzal (1997) (abbreviated RS-PDE and ZFV-PDE for convenience), and the lower bound results of Rogers and Shi (1995) and Chalasani, Jha, and Varikooty (1998) is presented in Table 5 with \( S_0 = 100 \), \( T = 1 \) and various choices of \( r \), \( v \) and \( K \). Our estimates of the continuous option values are based on price estimates with 26, 52 and 104 monitoring dates. There is excellent agreement in some instances: for example, when \( r = 0.15 \), \( v = 0.10 \) and \( K = 90 \), the option prices using the five methods are 15.398, 15.399, 15.404, 15.399, 15.398.

However, a more careful analysis of the option values reveals that the RS-PDE solution and C-LB are relatively less accurate to the other alternatives; the ZFV-PDE solution, R-LB and RNI yield price estimates that are practically identical in \textit{all} cases we consider. In this
respect, we find that the RS-PDE solution tends to be unstable for out-of-the-money options with low volatility while the C-LB performs \textit{badly} in generally, especially for in-the-money and out-of-the-money options with high volatility, \textit{despite the authors’ claim that their lower bounds are very accurate and tight} (in that the upper bounds often coincide with the lower bounds).

Therefore, taking the ZFV-PDE solutions as benchmark values, we conclude that our simple scheme for evaluating continuous option values performs rather well for all parameter values. We also note here that the ZFE-PDE solutions dominate the estimates of Alziary, Décamps, and Koehl (1997) and that our estimates compare favorably with the option values of Little and Pant (2000), who used a numerical moments approach to approximate the true distribution of the price average with the Johnson family of curves and in turn obtain price estimates.

The implication of our observation goes beyond a simple procedure for estimating the price of continuously monitored Asian options. The method of RNI can become inefficient when the number fixings $m$ gets increasingly large. However, the approximate formula $C_0^\alpha - \beta m^{-1}$, with the constants $C_0^\alpha$ and $\beta$ estimated using either Richardson’s extrapolation or least squares regression, allows us to estimate the price of an option with \textit{any} number of monitoring dates, provided we have the price estimates needed for either scheme to be implemented. As we have demonstrated, $C_0^\alpha$ and $\beta$ can be estimated using relatively small numbers of monitoring dates, making the amount of computational time reasonable.

7. DENSITY APPROXIMATION

With sufficient computational time, especially for high fixing frequencies, RNI is able to produce very accurate estimates of option prices. In situations where time-critical analysis of Asian options is necessary, it is feasible that RNI densities are computed ahead of time and stored for later retrieval in pricing problems. Another approach would involve adopting a parametric approximation to the true density of the arithmetic average for the sequence of lognormal random variables. There is numerical evidence to suggest that the density of an appropriately parametrized lognormal distribution offers a reasonably good approximation in cases associated with low volatility or short maturity. We propose a more general “mixed” density for this purpose and demonstrate that this mixed density offers a uniformly superior approximation over a wide range of option parameters.
7.1 Mixed Density Approximation

The main inadequacy of the lognormal approximation is that it fails to account for asymmetric tails in the true distribution of \( \log Y_m \). For this reason, the lognormal approximation fails for moderate to high volatility, as we demonstrated in Section 4.3; in particular, see Figure 2. In order to incorporate this feature into our approximation, as well as to retain tractability of the approximate density, we assume that \( \log Y_m \) is approximately distributed with the mixed density of the following form:

\[
n^*(x) = \begin{cases} 
    k(\nu_1, \nu_2, b_1, b_2) \exp \left[-\frac{1}{2} \left( \frac{a - x}{b_1} \right)^{\nu_1} \right], & x < a, \\
    k(\nu_1, \nu_2, b_1, b_2) \exp \left[-\frac{1}{2} \left( \frac{x - a}{b_2} \right)^{\nu_2} \right], & x \geq a,
\end{cases}
\]  

where \( \nu_2 < 2 < \nu_1 \) and \( k(\nu_1, \nu_2, b_1, b_2) = [b_1 \nu_1^{-1} 2^{1/\nu_1} \Gamma(1/\nu_1) + b_2 \nu_2^{-1} 2^{1/\nu_2} \Gamma(1/\nu_2)]^{-1} \). In (15), \( a \) is identified as the mode of the distribution, \( b_1 \) (resp. \( b_2 \)) as the scale parameter of the left (resp. right) tail, and \( \nu_1 \) (resp. \( \nu_2 \)) as the shape parameter of the left (resp. right) tail.

We are then able to write down the pricing formula for arithmetic average call options using the mixed density approximation. We denote by \( G_\alpha(\cdot) \) the standard Gamma distribution function with shape parameter \( \alpha \), given by

\[
G_\alpha(t) = \int_0^t \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \, dx,
\]

and let \( \bar{G}_\alpha(\cdot) := 1 - G_\alpha(\cdot) \). Recall from (2) that

\[
C_0 = e^{-rT} E[S_0 Y_m - K]^+.
\]

**Proposition 2.** Assume that the density of \( \log Y_m \) is given by (15). Then the initial value of an arithmetic average call option is given by

\[
C_0 = kS_0 e^{-rT} \left\{ e^a \left[ \sum_{i=0}^{\infty} (-1)^i \eta_i \bar{G}_{\eta_i}(\xi_1) + \sum_{i=0}^{\infty} \eta_2 \right] - K^* \left[ u_{01} \bar{G}_{u_{01}}(\xi_1) + u_{02} \right] \right\} \quad \text{if } \log K^* < a,
\]

\[
(16a)
\]

\[
C_0 = kS_0 e^{-rT} \left\{ e^a \left[ \sum_{i=0}^{\infty} u_{i2} \bar{G}_{\eta_i}(\xi_2) - K^* u_{02} \bar{G}_{u_{02}}(\xi_2) \right] \right\} \quad \text{if } \log K^* \geq a,
\]

\[
(16b)
\]

where \( k \) is short for \( k(\nu_1, \nu_2, b_1, b_2) \) and

\[
K^* = \frac{K}{S_0}, \quad \xi_1 = \frac{1}{2} \left( \frac{a - \log K^*}{b_1} \right)^{\nu_1}, \quad \xi_2 = \frac{1}{2} \left( \frac{\log K^* - a}{b_2} \right)^{\nu_2},
\]

\[
\eta_{ij} = \frac{i + 1}{\nu_j}, \quad u_{ij} = \frac{b_j^{\nu_j} \Gamma(\eta_{ij}) 2^{\eta_{ij}}}{i!}, \quad i = 0, 1, 2, \ldots, \quad j = 1, 2.
\]
Proof of Proposition 2. Following the remark just before Proposition 2, we have

\[
C_0 = S_0 e^{-\tau T} \left[ \int_{\log K^*}^{\infty} e^x n^*(x) \, dx - K^* \int_{\log K^*}^{\infty} n^*(x) \, dx \right]
\]

\[
= S_0 e^{-\tau T} \left[ e^a \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\log K^*}^{\infty} (x-a)^i n^*(x) \, dx - K^* \int_{\log K^*}^{\infty} n^*(x) \, dx \right].
\]  \hspace{1cm} (17)

We now consider the integrals \( \int_{\log K^*}^{\infty} (x-a)^i n^*(x) \, dx \) (\( i = 0, 1, 2, \ldots \)) when \( \log K^* < a \) and when \( \log K^* \geq a \). We will make use of the following substitutions where appropriate:

\[
u_1 = 1 \quad \text{and} \quad \nu_2 = 1 \quad \text{for} \quad a \quad \text{and} \quad u = \frac{1}{2} \left( \frac{x-a}{b_1} \right)^{\nu_1}
\]

In the case \( \log K^* < a \),

\[
\int_{\log K^*}^{\infty} (x-a)^i n^*(x) \, dx = k \int_{\log K^*}^{\infty} (x-a)^i e^{-\frac{1}{2} \left( \frac{x-a}{b_1} \right)^2} \, dx + k \int_{a}^{\infty} (x-a)^i e^{-\frac{1}{2} \left( \frac{x-a}{b_2} \right)^2} \, dx
\]

\[
= \frac{kb_1}{\nu_1} (-1)^i b_1^{2\nu_1} \int_0^{\xi_1} e^{-u} u^{\nu_1-1} \, du + \frac{kb_2}{\nu_2} b_2^{2\nu_2} \int_0^{\xi_2} e^{-u} u^{\nu_2-1} \, du
\]

\[
= \frac{kb_1}{\nu_1} (-1)^i b_1^{2\nu_1} \Gamma(\eta_1) 2^{\eta_1} G_{\eta_1}(\xi_1) + \frac{kb_2}{\nu_2} b_2^{2\nu_2} \Gamma(\eta_2) 2^{\eta_2} G_{\eta_2}(\xi_2), \hspace{1cm} (18a)
\]

following the definition of \( G_{\alpha}(\cdot) \) and the fact that \( G_{\alpha}(t) \rightarrow 1 \) as \( t \rightarrow \infty \).

We can similarly deal with the other case \( \log K^* \geq a \) by noting that

\[
\int_{\log K^*}^{\infty} (x-a)^i n^*(x) \, dx = k \int_{\log K^*}^{\infty} (x-a)^i e^{-\frac{1}{2} \left( \frac{x-a}{b_2} \right)^2} \, dx
\]

\[
= \frac{kb_2}{\nu_2} b_2^{2\nu_2} \int_0^{\xi_2} e^{-u} u^{\nu_2-1} \, du = \frac{kb_2}{\nu_2} b_2^{2\nu_2} \Gamma(\eta_2) 2^{\eta_2} G_{\eta_2}(\xi_2). \hspace{1cm} (18b)
\]

Equations (16a)–(16b) therefore follow by substituting (18a)–(18b) into (17).

7.2 Estimation of Parameters

We outline the procedure for estimating the parameters \( (a, \nu_1, \nu_2, b_1, b_2) \) and normalizing constant \( k(\nu_1, \nu_2, b_1, b_2) \). In order to make use of the hypothesized density (15), we transform the RNI density \( f_m(x) \) of \( Y_m \) to the “log-scale” density \( n_m(x) \) of \( \log Y_m \), via \( n_m(x) = e^x f_m(e^x) \).

First, we obtain an estimate \( \hat{a} \) of \( a \) using the location of the mode in the log-scale density. This is achieved through a quadratic fit to the log-scale density in the neighborhood of the mode. Denoting by \( (x_L, y_L), (x_0, y_0) \) and \( (x_R, y_R) \) the three points in this neighborhood, where \( y_i = n_m(x_i) \) for \( i = L, 0, R \), we have

\[
\hat{a} = \frac{1}{2} \left[ \frac{(x_0^2 - x_L^2)(y_0 - y_R) + (x_R^2 - x_0^2)(y_0 - y_L)}{(x_0 - x_L)(y_0 - y_R) + (x_R - x_0)(y_0 - y_L)} \right]. \hspace{1cm} (19)
\]
From this we also obtain an estimate $\hat{k}$ for the normalizing constant $k(\nu_1, \nu_2, b_1, b_2)$:

$$\hat{k} = y_0 + (x_0 - \hat{a})^2 \left[ \frac{y_0 - y_R}{(x_R - \hat{a})^2 - (x_0 - \hat{a})^2} \right].$$

(20)

Next, it can be shown from (15) that

$$\log \left\{ 2 \log \left[ \frac{k}{n^*(x)} \right] \right\} = \begin{cases} \nu_1 \log(a - x) - \nu_1 \log b_1, & x < a, \\ \nu_2 \log(x - a) - \nu_2 \log b_2, & x \geq a. \end{cases}$$

Subject to the accuracy of the estimates $(\hat{a}, \hat{k})$, we expect the following two plots based on the log-scale density to yield straight lines:

- **Line $\ell_1$:** Plot $\log \left\{ 2 \log \left[ \frac{k}{n^*(x)} \right] \right\}$ against $\log(\hat{a} - x)$ for $x < \hat{a}$;

- **Line $\ell_2$:** Plot $\log \left\{ 2 \log \left[ \frac{k}{n^*(x)} \right] \right\}$ against $\log(x - \hat{a})$ for $x \geq \hat{a}$.

From the least-squares estimates of the gradients and vertical intercepts of the lines $\ell_1$ and $\ell_2$, denoted by $(\hat{m}_1^{LS}, \hat{c}_1^{LS})$ and $(\hat{m}_2^{LS}, \hat{c}_2^{LS})$ respectively, we obtain the estimates $(\hat{\nu}_1, \hat{\nu}_2, \hat{b}_1, \hat{b}_2)$ of $(\nu_1, \nu_2, b_1, b_2)$:

$$\hat{\nu}_1 = \hat{m}_1^{LS}, \quad \hat{b}_1 = \exp(-\hat{c}_1^{LS}/\hat{\nu}_1); \quad \hat{\nu}_2 = \hat{m}_2^{LS}, \quad \hat{b}_2 = \exp(-\hat{c}_2^{LS}/\hat{\nu}_2).$$

(21)

As an illustration, we report in Table 6 the parameter estimates (19)–(21) for market parameters (12a) and $v = 0.05, 0.10, 0.30, 0.50$. Since the mixed density (15) is after all an approximation of the true density of $\log Y_m$, we cannot expect to have an exact linear fit when computing the estimates (21) using regression. As a guide, we leave out the first and last 16% of the points in line $\ell_1$ when estimating $\nu_1$ and $b_1$; when estimating $\nu_2$ and $b_2$, we leave out the first and last 16% (resp. 20% and 24%) of the points in $\ell_2$ for high (resp. moderate and low) volatility. It is clear from these estimates that the density of $\log Y_m$ deviates more from symmetry as the volatility increases, which reinforces the inadequacy of the lognormal approximation.

**INSERT TABLE 6 ABOUT HERE**

### 7.3 Results and Discussion

In Section 4.3, we argued that the lognormal approximation for the true distribution of $\log Y_m$ is inadequate by using quantile-plots of simulated $\log Y_m$ against the hypothesized normal distribution. We will rely on the same procedure to demonstrate that our mixed density approximation (15) presents significant improvements over the lognormal approximation.
For a start, we work out the quantiles of the mixed density. Along the lines of the proof of Proposition 2, it can be shown that 
\[ N^*(x) = \begin{cases} \frac{k_b}{\nu_1} \Gamma(1/\nu_1) 2^{1/\nu_1} G_1/\nu_1 \left[ \frac{1}{2} \left( \frac{a-x}{b_1} \right)^{\nu_1} \right], & x < a, \\ \frac{k_b}{\nu_1} \Gamma(1/\nu_1) 2^{1/\nu_1} + \frac{k_b}{\nu_2} \Gamma(1/\nu_2) 2^{1/\nu_2} G_1/\nu_2 \left[ \frac{1}{2} \left( \frac{x-a}{b_2} \right)^{\nu_2} \right], & x \geq a, \end{cases} \]
The p-quantile \( x_p \) of the mixed density (15) can then be obtained as the solution of \( N^*(x_p) = p \):
\[ x_p = \begin{cases} a - b_1 \left\{ 2G_1^{-1/\nu_1} \left[ 1 - \frac{p\nu_1}{k_b \Gamma(1/\nu_1) 2^{1/\nu_1}} \right] \right\}^{1/\nu_1}, & p < \frac{k_b}{\nu_1} \Gamma(1/\nu_1) 2^{1/\nu_1}, \\ a + b_2 \left\{ 2G_1^{-1/\nu_2} \left[ \frac{p\nu_2}{k_b \Gamma(1/\nu_2) 2^{1/\nu_2}} - \frac{\nu_2 b_1 \Gamma(1/\nu_1) 2^{1/\nu_1}}{\nu_1 b_2 \Gamma(1/\nu_2) 2^{1/\nu_2}} \right] \right\}^{1/\nu_2}, & p \geq \frac{k_b}{\nu_1} \Gamma(1/\nu_1) 2^{1/\nu_1}. \]
Based on a 10000 MC simulation series of \( \log Y_m \), a quantile plot of the simulated \( \log Y_m \) against the mixed density approximation would consist of plotting the \( i \)th ordered \( \log Y_m \) against \( x_p \), where \( p_i = (i - 0.5)/10000 \), for \( i = 1, \ldots, 10000 \). We evaluate the \( x_p \)'s using the estimated parameters (19)–(21). Quantile-plots for market parameters (12a) and \( v = 0.30, 0.50 \) are shown in Figure 4. It is evident that the points lie much closer to the identity line than they did when we used the lognormal approximation; cf. left panels of Figures 2(b)–(c). We also plot the approximate density \( f^*(x) \) of \( Y_m \) on the same axes as the RNI density for these two cases in Figure 4, where \( f^*(x) = x^{-1} n^*(\log x) \). The plots show that our mixed density approximation is better able to capture the tails of the true distribution of \( Y_m \) accurately, even with moderate to high volatility. In fact, the approximate density and the RNI density are virtually indistinguishable in our plots.

Finally, we show that the mixed density approximation leads to accurate option prices against MC simulation. Using the setting of Table 1, i.e., with parameters (12a), we obtain the price estimates of the arithmetic average call options through the pricing formulas (16a)–(16b), truncating the infinite series after 100 terms. The results are summarized in Table 1. It is clear that the mixed density approximation leads to reasonably accurate option prices.

8. CONCLUSION

We have presented a unified approach for the pricing of both discrete and continuous Asian options. The method of recursive numerical integration is applicable to the pricing of both
fixed strike and floating strike discrete Asian options. Straightforward extensions of the basic method allow us to price the options other than at their inception, such as prior to the averaging period, or into the averaging period. By virtue of the put-call parity relation (10), all put options are also covered. The key advantage of this method, other than computational efficiency, is that the accuracy of the RNI price estimates are not sensitive to changes in the volatility and/or time to maturity.

More importantly, we demonstrate that there exists a quantitative relationship between the price of an Asian option and the number of fixings. This relationship can be capitalized to yield option prices in the case of continuous fixings. One approach calls for the use of RNI price estimates in the evaluation of the continuous option prices, either via Richardson’s extrapolation or by regression. Along with the continuous option values, we also solve for the relationship between option values and fixing frequency, thereby enabling us to price Asian options with any number of fixings. Alternatively, the RNI densities are themselves useful for the determination of a mixed density approximation for the true density of the discrete price average, thereby leading to new approximations for Asian option values.
APPENDIX

In this appendix we will derive and justify the appropriate "truncation points" used in our implementation of RNI in Section 4 (see Proposition 1). Specifically, we will analyze the recursive densities in two respects. First, for each density, we seek truncation points \( x \) and \( y \) that make it negligible outside the interval \((x, y)\). Requiring that the coverage probability exceeds a certain predetermined lower bound governs the "effective range" of the support of a density that will be retained, and the corresponding threshold value below which the density will be discarded.

**Distributional Properties of \( Y_1 \)**

Given \( \varepsilon \), \( f_1(x) \leq \varepsilon \) whenever \( x \) lies outside the interval \((x_1, \overline{x}_1)\), where \( x_1 = h_1(1 + \exp(-\sigma B_1 - (\sigma^2 - \mu))) \) and \( \overline{x}_1 = h_1(1 + \exp(\sigma B_1 - (\sigma^2 - \mu))) \), with \( B_1^2 = \sigma^2 - 2\mu - 2\log(\varepsilon h_1\sqrt{2\pi}) \geq 0.7 \). Moreover,

\[
P(x_1 \leq Y_1 \leq \overline{x}_1) = N(B_1 - \sigma) - N(-B_1 - \sigma).
\]

If it is desired that the last probability exceeds some \( \alpha \), it suffices for us to seek the smallest \( B_1 \) such that \( N(B_1 - \sigma) > (1 + \alpha)/2 \). We then have the appropriate values of \( B_1 \) and \( \varepsilon \), and hence of \( x_1 \) and \( \overline{x}_1 \), as stated in Proposition 1(a):

\[
B_1^*(\alpha) = \sigma + N^{-1}((1 + \alpha)/2) \quad \text{and} \quad \varepsilon_1^*(\alpha) = (\sigma h_1\sqrt{2\pi})^{-1} \exp((\sigma^2 - 2\mu - B_1^*(\alpha)^2)/2),
\]

\[
x_1^*(\alpha) = h_1 + h_1 \exp(-\sigma B_1^*(\alpha) - (\sigma^2 - \mu)) \quad \text{and} \quad \overline{x}_1^*(\alpha) = h_1 + h_1 \exp(\sigma B_1^*(\alpha) - (\sigma^2 - \mu)).
\]

**Distributional properties of \( Y_i \) \((i \geq 2)\)**

Let \( \psi_i(x, y) \) be as defined in (11). We first view \( \psi_i(x, y) \), for fixed \( i \) and \( x \), as a function of \( y \) in \((h_{i-1}, \infty)\). Given \( \varepsilon \), \( \psi_i(x, y) \leq \varepsilon \) whenever \( y \leq y_{i-1}(x) \) or \( y \geq \overline{y}_{i-1}(x) \), where \( y_{i-1}(x) = (h_{i-1}/h_i)(x - h_i) \exp(-\sigma D_i(x) - \mu) \) and \( \overline{y}_{i-1}(x) = (h_{i-1}/h_i)(x - h_i) \exp(\sigma D_i(x) - \mu) \), with \( D_i(x)^2 = -2\log(\sigma \varepsilon (x - h_i)\sqrt{2\pi}) \geq 0.8 \).

We say \((\underline{x}_i, \overline{x}_i)\) is an \( \alpha \)-coverage range of \( f_i \) if \( \int_{\underline{x}_i}^{\overline{x}_i} f_i(y) \, dy \geq \alpha^i \). Suppose we have obtained \( f_j \) \((j = 1, \ldots, i - 1; \ i \geq 2)\) in their respective \( \alpha \)-coverage ranges \((\underline{x}_j, \overline{x}_j)\). We can rewrite (7b) in the following approximate form:

\[
f_i(x) \approx \int_{\underline{x}_{i-1}}^{\overline{x}_{i-1}} \psi_i(x, y) f_{i-1}(y) \, dy.
\]

Let \( \underline{x}_i \) is the largest \( x \) such that \( \psi_i(x, y) \leq \varepsilon \) for all \( y \geq \underline{x}_{i-1} \) and \( \overline{x}_i \) is the smallest \( x \) such that \( \psi_i(x, y) \leq \varepsilon \) for all \( y \leq \overline{x}_{i-1} \). Then it is easy to see that \( f_i(x) \leq \varepsilon \) whenever \( x \leq \underline{x}_i \) or
Since \( x_i \) is the largest \( x \) such that \( y_i-1(x) \leq x_i-1 \) and \( \pi_i \) is the smallest \( x \) such that \( y_i-1(x) \geq \pi_i-1 \), we obtain
\[
\bar{x}_i = h_i + h_i \bar{x}_i-1 \bar{h}_i-1 \exp(-\sigma B_i - (\sigma^2 - \mu)) \quad \text{and} \quad \bar{\pi}_i = h_i + h_i \bar{\pi}_i-1 \bar{h}_i-1 \exp(\sigma B_i - (\sigma^2 - \mu)),
\]
where
\[
B_i^2 = \sigma^2 - 2\mu - 2 \log(\sigma \varepsilon \bar{x}_i \sqrt{2\pi}) - 2 \log(h_i/h_i-1) \geq 0,
\]
\[
\bar{B}_i^2 = \sigma^2 - 2\mu - 2 \log(\sigma \varepsilon \bar{\pi}_i \sqrt{2\pi}) - 2 \log(h_i/h_i-1) \geq 0.
\]
Next, we seek the coverage probability of the interval \((\bar{x}_i, \bar{\pi}_i)\). We find that
\[
P(\bar{x}_i \leq Y_i \leq \bar{\pi}_i)
\geq \int_{\bar{x}_i-1}^{\bar{\pi}_i-1} \left\{ N \left[ \sigma^{-1} \log \left( \frac{h_i-1(\bar{x}_i - h_i)}{y h_i e^\mu} \right) \right] - N \left[ \sigma^{-1} \log \left( \frac{h_i-1(\bar{\pi}_i - h_i)}{y h_i e^\mu} \right) \right] \right\} f_i-1(y) dy
\geq \int_{\bar{x}_i-1}^{\bar{\pi}_i-1} \left\{ N \left[ \sigma^{-1} \log \left( \frac{h_i-1(\bar{x}_i - h_i)}{\bar{\pi}_i-1 h_i e^\mu} \right) \right] - N \left[ \sigma^{-1} \log \left( \frac{h_i-1(\bar{\pi}_i - h_i)}{\bar{\pi}_i-1 h_i e^\mu} \right) \right] \right\} f_i-1(y) dy
= \left[ N(\bar{B}_i - \sigma) - N(-\bar{B}_i - \sigma) \right] \left( \int_{\bar{x}_i-1}^{\bar{\pi}_i-1} f_i-1(y) dy \right)
> \alpha_i^{-1} \left[ N(\bar{B}_i - \sigma) - N(-\bar{B}_i - \sigma) \right],
\]
since \((\bar{x}_i-1, \bar{\pi}_i-1)\) is an \( \alpha \)-coverage range of \( f_i-1 \). This coverage probability will therefore exceed \( \alpha_i \) if \( \varepsilon \) (hence \( B_i \) and \( \bar{B}_i \)) is determined such that \( N(\bar{B}_i - \sigma) - N(-\bar{B}_i - \sigma) > \alpha \). Noting that \( \bar{x}_i-1 < \bar{\pi}_i-1 \), so \( \bar{B}_i > \bar{B}_i \), it suffices to seek the smallest \( \bar{B}_i \) such that \( 2N(\bar{B}_i - \sigma) - 1 > \alpha \). The appropriate values of \( \bar{B}_i, B_i \) and \( \varepsilon \), and hence of \( x_i \) and \( \pi_i \) as given in Proposition 1(b), now follows:
\[
\bar{B}_i^*(\alpha) = \sigma + N^{-1}((1 + \alpha)/2) \quad \text{and} \quad B_i^*(\alpha) = \sqrt{\bar{B}_i^*(\alpha)^2 + 2 \log(\bar{x}_i-1/\bar{x}_i-1)},
\]
\[
\varepsilon_i^*(\alpha) = h_i-1(\sigma \bar{x}_i-1 h_i \sqrt{2\pi})^{-1} \exp(((\sigma^2 - 2\mu - B_i^*(\alpha)^2)/2);
\]
\[
\bar{x}_i^*(\alpha) = h_i + h_i \bar{x}_i-1 \bar{h}_i-1 \exp(-\sigma B_i^*(\alpha) - (\sigma^2 - \mu)),
\]
\[
\bar{\pi}_i^*(\alpha) = h_i + h_i \bar{\pi}_i-1 \bar{h}_i-1 \exp(\sigma B_i^*(\alpha) - (\sigma^2 - \mu)).
\]
REFERENCES


NOTES

1 We have chosen equally spaced monitoring dates for two reasons. This choice of fixings simplifies our notation and allows us to focus on ideas underlying the construction of the recursive sequence in Section 3. We leave it to the reader to make the necessary and direct modifications to deal with nonuniform fixings. More importantly, the case of equally spaced fixings leads naturally to the case of continuous fixings as \( m \to \infty \), which we investigate in Section 6.

2 We write \( A \overset{d}{=} B \) to mean that \( A \) and \( B \) have the same distribution (law).

3 To estimate the price of a discretely monitored option, we modify the variance reduction procedure in Kemna and Vorst (1990) slightly. Specifically, we estimate \( E[(A_m - K)^+ - (G_m - K)^+] \) from sample path simulations and add this estimate to the theoretical value of \( E[G_m - K]^+ \) given by (4), instead of \( E[G_T - K]^+ \), to obtain an estimate of \( E[A_m - K]^+ \).

4 The true distribution of \( \log Y_m \) is based on a 10000 MC simulation series and is standardized with the assumed mean \( \alpha_m \) and standard deviation \( \beta_m \).

5 In analogy to Kemna and Vorst (1990) we generate both the arithmetic average \( A_m \) and the geometric average \( G_m \) in simulation. We would obtain \( E[S_m - A_m]^+ \) as the sum of two terms: an estimate of \( E[(S_m - A_m)^+ - (S_m - G_m)^+] \) through MC simulation, and the theoretical value of \( E[S_m - G_m]^+ \) given by (5).

6 We emphasize that quasi-analytic approaches based on continuous fixings cannot be easily adapted to estimate the prices of discretely monitored Asian options.

7 If this positivity condition is not satisfied, then \( f_1(x) \leq \varepsilon \) for all \( x > h_1 \). (That is, the given \( \varepsilon \) is too “large.”)

8 This positivity condition is equivalent to \( x < h_i + (\sigma \varepsilon \sqrt{2i})^{-1} \). If it is not satisfied, then \( \psi_i(x, y) \leq \varepsilon \) for all \( y > h_{i-1} \).

9 The values of \( x \) that would make the density \( f_i(x) \) small are found by requiring that \( f_{i-1}(y) \) convolves either with the extreme right tail of \( \psi_i(x, y) \) or with the extreme left tail of \( \psi_i(x, y) \). Both \( \underline{x}_i \) and \( \overline{x}_i \) are well-defined when \( \sigma^2 - 2\mu - 2 \log(\sigma \varepsilon h_i \sqrt{2i}) > 0 \) and \( \mu + \log(\sigma \varepsilon h_i \sqrt{2i}) < 0 \).

10 If the positivity conditions are not satisfied, then the chosen \( \varepsilon \) is again too large.
TABLE 1. Initial values of discrete fixed strike Asian call options with \( S_0 = 100 \), \( r = 0.09 \), \( T = 1 \) and \( m = 52 \), through MC simulation, using RNI and via the mixed density approximation (MDA). The MC simulation results are based on Levy and Turnbull (1992).

<table>
<thead>
<tr>
<th>( K )</th>
<th>( v = 0.05 )</th>
<th>( v = 0.10 )</th>
<th>( v = 0.30 )</th>
<th>( v = 0.50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>RNI</td>
<td>MDA</td>
<td>MC</td>
</tr>
<tr>
<td>100</td>
<td>4.31</td>
<td>4.308</td>
<td>4.91</td>
<td>4.911</td>
</tr>
<tr>
<td>105</td>
<td>0.95</td>
<td>0.954</td>
<td>2.06</td>
<td>2.064</td>
</tr>
<tr>
<td>110</td>
<td>–</td>
<td>0.051</td>
<td>–</td>
<td>0.624</td>
</tr>
</tbody>
</table>

TABLE 2. Initial values of discrete fixed strike Asian call options with \( S_0 = 100 \) and \( r = 0.10 \), through MC simulation, via the FSG algorithm and using RNI. The FSG results are based on Barraquand and Pudet (1996).

<table>
<thead>
<tr>
<th>( T )</th>
<th>( m )</th>
<th>( K )</th>
<th>( v = 0.10 )</th>
<th>( v = 0.20 )</th>
<th>( v = 0.40 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>MC</td>
<td>FSG</td>
<td>RNI</td>
</tr>
</tbody>
</table>
\hline
|       | 100   | 1.848 | 1.869 | 1.845 | 2.926 | 2.960 | 2.921 | 5.159 | 5.218 | 5.146 |
|       | 105   | 0.147 | 0.151 | 0.146 | 0.944 | 0.966 | 0.939 | 3.054 | 3.106 | 3.043 |
|       | 105   | 0.711 | 0.727 | 0.706 | 2.197 | 2.241 | 2.191 | 5.359 | 5.444 | 5.344 |
|       | 100   | 5.245 | 5.279 | 5.241 | 7.037 | 7.079 | 7.021 | 11.111| 11.213| 11.090|
TABLE 3. Initial values of discrete fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $\nu = 0.30$ and $T = 1$ using RNI, for different values of $m$. Numbers in parentheses show increase in option price over the previous value. Continuous option values are obtained using the LS-6 scheme (see text and Table 4).

<table>
<thead>
<tr>
<th>$K$</th>
<th>26</th>
<th>52</th>
<th>104</th>
<th>208</th>
<th>416</th>
<th>832</th>
<th>$\infty^{\text{LS-6}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>14.9434</td>
<td>14.9629 (0.0195)</td>
<td>14.9728 (0.0099)</td>
<td>14.9779 (0.0051)</td>
<td>14.9803 (0.0024)</td>
<td>14.9815 (0.0012)</td>
<td>14.9827</td>
</tr>
<tr>
<td>95</td>
<td>11.6062</td>
<td>11.6306 (0.0244)</td>
<td>11.6427 (0.0121)</td>
<td>11.6487 (0.0060)</td>
<td>11.6518 (0.0031)</td>
<td>11.6533 (0.0015)</td>
<td>11.6548</td>
</tr>
<tr>
<td>100</td>
<td>8.7742</td>
<td>8.8006 (0.0284)</td>
<td>8.8143 (0.0137)</td>
<td>8.8210 (0.0067)</td>
<td>8.8244 (0.0034)</td>
<td>8.8260 (0.0016)</td>
<td>8.8277</td>
</tr>
<tr>
<td>105</td>
<td>6.4631</td>
<td>6.4899 (0.0268)</td>
<td>6.5031 (0.0131)</td>
<td>6.5100 (0.0069)</td>
<td>6.5134 (0.0034)</td>
<td>6.5151 (0.0017)</td>
<td>6.5167</td>
</tr>
<tr>
<td>110</td>
<td>4.6459</td>
<td>4.6704 (0.0245)</td>
<td>4.6832 (0.0128)</td>
<td>4.6895 (0.0063)</td>
<td>4.6927 (0.0032)</td>
<td>4.6942 (0.0015)</td>
<td>4.6957</td>
</tr>
</tbody>
</table>

TABLE 4. Initial values $C_0^*$ and coefficient $\beta$ of continuous fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $\nu = 0.30$ and $T = 1$, estimated using Richardson’s extrapolation (RE-104 and RE-416) or least squares regression (LS-3 and LS-6).

<table>
<thead>
<tr>
<th>$K$</th>
<th>RE-104</th>
<th>RE-416</th>
<th>LS-3</th>
<th>LS-6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_0^*$</td>
<td>$\beta$</td>
<td>$C_0^*$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>90</td>
<td>14.9829</td>
<td>1.04465</td>
<td>14.9828</td>
<td>1.00608</td>
</tr>
<tr>
<td>95</td>
<td>11.6547</td>
<td>1.24816</td>
<td>11.6547</td>
<td>1.21612</td>
</tr>
<tr>
<td>100</td>
<td>8.8276</td>
<td>1.37946</td>
<td>8.8277</td>
<td>1.35312</td>
</tr>
<tr>
<td>105</td>
<td>6.5169</td>
<td>1.43085</td>
<td>6.5168</td>
<td>1.40900</td>
</tr>
<tr>
<td>110</td>
<td>4.6957</td>
<td>1.29682</td>
<td>4.6958</td>
<td>1.27839</td>
</tr>
</tbody>
</table>
TABLE 5. Initial values of continuous fixed strike Asian call options with $S_0 = 100$ and $T = 1$, through PDE solution, via the R-LB and C-LB, and using RNI. The PDE solutions (headed by author initials) are due to Rogers and Shi (1995) and Zvan, Forsyth, and Vetzal (1997); the ZFV results are available for $r = 0.15$ only. The R-LB and C-LB results are based on Rogers and Shi (1995) and Chalasani, Jha, and Varikooty (1998) respectively.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$K$</th>
<th>$r = 0.05$</th>
<th>$r = 0.09$</th>
<th>$r = 0.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>RS</td>
<td>C-LB</td>
<td>R-LB</td>
</tr>
<tr>
<td>0.05</td>
<td>95</td>
<td>7.157</td>
<td>7.178</td>
<td>7.178</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>0.439</td>
<td>0.309</td>
<td>0.337</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>0.359</td>
<td>0.306</td>
<td>0.331</td>
</tr>
</tbody>
</table>
TABLE 6. Parameter estimates of the mixed density approximation for \( r = 0.09 \), \( T = 1 \) and \( m = 52 \).

<table>
<thead>
<tr>
<th>( v )</th>
<th>Parameter estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{a} )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.04467</td>
</tr>
<tr>
<td>0.10</td>
<td>0.04261</td>
</tr>
<tr>
<td>0.30</td>
<td>0.02080</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.02239</td>
</tr>
</tbody>
</table>

FIGURE 1. Initial values of discrete fixed strike Asian call options with \( S_0 = 100 \), \( r = 0.09 \), \( T = 1 \), \( m = 52 \) and \( K = 100 \), through MC simulation (boxplots) and using RNI (solid lines). Summary statistics of 1000 simulation runs for each volatility are listed under the boxplots.
FIGURE 2. The lognormal distribution as an approximation of the true distribution of the arithmetic average, for parameters $r = 0.09$, $T = 1$ and $m = 52$. (left) Quantile-plots against the standard normal distribution. (right) Comparison of lognormal densities (dotted lines) and RNI densities.

(a) $v = 0.10$ (low volatility): $\alpha_m = 0.04366$, $\beta_m = 0.05817$

(b) $v = 0.30$ (moderate volatility): $\alpha_m = 0.03002$, $\beta_m = 0.17510$

(c) $v = 0.50$ (high volatility): $\alpha_m = 0.00217$, $\beta_m = 0.29387$
FIGURE 3. Initial values of discrete floating strike Asian call options with $S_0 = 100$ and $r = 0.10$, through MC simulation (boxplots), via the FSG algorithm and using RNI (dotted lines). The FSG results are based on Barraquand and Pudet (1996).
FIGURE 4. The mixed density approximation of the true distribution of the arithmetic average, for parameters $r = 0.09$, $T = 1$, and $m = 52$. (left) Quantile-plots against the mixed density approximation. (right) Comparison of approximate densities (dotted lines) and RNI densities.