Exam Questions (Special Term 2001)

Where necessary, you may assume that the Black-Scholes environment prevails so that the asset price is lognormally distributed. That is, $S_t = S_0 e^{X}$, where $X$ is normally distributed with mean $(r - q - \sigma^2/2)t$ and standard deviation $\sigma \sqrt{t}$.

1. Consider the following path-dependent options:

<table>
<thead>
<tr>
<th>Option type</th>
<th>Payoff at maturity</th>
<th>Risk-neutral pricing formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Lookback option</td>
<td>$M_T - S_T$</td>
<td>$LB = e^{-rT} E[M_T - S_T]$</td>
</tr>
<tr>
<td>(ii) Lookforward option</td>
<td>$M_T - S_0$</td>
<td>$LF = e^{-rT} E[M_T - S_0]$</td>
</tr>
<tr>
<td>(iii) Hindsight option</td>
<td>$\max{0, M_T - K}$</td>
<td>$HS = e^{-rT} E[\max{0, M_T - K}]$</td>
</tr>
</tbody>
</table>

Here, $M_T$ denotes the maximum observed asset price in the interval $[0, T]$. All three options use the same fixing frequency for $M_T$. **True or false:** (a) $LB > LF$; (b) $LF > HS$. Explain your answers.

2. A **participating option** is an option which changes the rate of participation in a price or rate movement once a prespecified price level is reached. Consider a cash-settled call option on the ABC stock index which gives 100% participation from a strike at-the-money up to the point at which the index has moved up 20%. Then further participation is limited to 50%. The payoff of such a participating option is illustrated below (not drawn to scale):

(i) By breaking down the participating option into its constituent parts, or otherwise, suggest how you would statically hedge a short position in the participating option.

(ii) By studying the constituent parts you obtained in part (i), or otherwise, obtain an expression for the value of the participating option one year from expiration.

3. (a) An outside barrier option pays off like a vanilla option on an asset under the condition that a second asset price has (or has not) breached a barrier. Specifically, an **outside down-and-out call option** pays $\max\{0, S_1(T) - K\}$ at maturity provided $S_2(t_i) \geq H$ on all fixing dates $t_1, t_2, \ldots, t_n$ in $[0, T]$.

Outline the steps involved when using Monte Carlo simulation to price an outside down-and-out call option. You may assume that $t_n = T$. 

1
(b) Consider another option that pays $\max\{0, S_1(T) - K\}$ at maturity provided $S_2(T) \geq H$.

(i) **True or false:** This option is an outside barrier option. Explain your answer.
(ii) Discuss the effect of correlation between $S_1(T)$ and $S_2(T)$ on the price of this option.
(iii) This option is priced at $2.55$ for the following parameters:

\[
\begin{align*}
& r = 8\%, \quad q_1 = 3\%, \quad \sigma_1 = 20\%, \quad K = 40, \quad S_1 = 40, \\
& \rho = 50\%, \quad q_2 = 5\%, \quad \sigma_2 = 30\%, \quad H = 35, \quad S_2 = 40, \quad T - t = 0.5.
\end{align*}
\]

By using an appropriate parity condition, or otherwise, obtain the price of an option that pays $\max\{0, S_1(T) - K\}$ at maturity provided $S_2(T) \leq H$.

4. An example of a payoff-modified option is the log option, whose payoff at maturity is

\[
\max\{0, \ln(S_T/K)\} = \max\{0, \ln S_T - \ln K\}, \quad \text{where } K \text{ is the strike price.}
\]

(i) Sketch the payoff profile of the log option and suggest a suitable application.
(ii) When the time to maturity is $\tau = T - t$, show that the value of log option is given by

\[
C = \sigma \sqrt{\tau} e^{-\frac{r\tau}{2}} \left[n(d_2) + d_2 N(d_2)\right], \quad \text{where } d_2 = \frac{\ln(S/K) + (r - q - \sigma^2/2)\tau}{\sigma \sqrt{\tau}}.
\]

[Hint: You may use the result $\int z n(z) dz = -n(z) + \text{constant}$.]

(iii) Calculate the price of the log option for the following parameters:

\[
\begin{align*}
& r = 8\%, \quad q = 3\%, \quad \sigma = 20\%, \quad K = 40, \quad S = 40, \quad \tau = 0.5.
\end{align*}
\]