Pricing and Hedging Asian Options: 
A Recursive Integration Approach

Lim Tiong Wee 
Dept. of Statistics and Appl. Prob. 
National University of Singapore

Master of Financial Engineering 
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http://www.stat.nus.edu.sg/~stalimtw/
Agenda

• Features of Asian options
• Survey of methods for pricing Asian options
• Recursive integration approach: fixed strike Asian options
  ○ Sequence of recursive definitions for arithmetic average
  ○ Details of recursive numerical integration
  ○ Numerical results and comparison
• Extension to floating strike Asian options
• Frequency of fixings and convergence of option values
  ○ Motivation using the geometric average
  ○ Extrapolation results for the arithmetic average
• Conclusion and discussion
Features of Asian Options

- Provision for early exercise: European-style vs. American-style.
- Type of strike: fixed strike vs. floating strike
  - Fixed strike: payoff = difference between price average and predetermined strike price (average rate option)
  - Floating strike: payoff = difference between spot price and price average (average strike option)
- Type of averaging: arithmetic average vs. geometric average
- Type of fixing: discrete fixing vs. continuous fixing
  - Discrete fixing: prices collected on discrete set of dates
  - Continuous fixing: prices collected over continuous interval of time
Survey of Methods

- Exact solution — Turnbull and Wakeman (JFQA '91) for geometric average options
- Monte Carlo simulation — Kemna and Vorst (JBF '90) with variance reduction
- Density approximation — Carverhill and Clewlow (Risk '90) using Fourier transform; Turnbull and Wakeman (JFQA '91) and Levy (JIMF '92) using lognormal approximation
- Binomial tree — Barraquand and Pudet (MF '96)
- Finite difference — Zvan, Forsyth, and Vetzal (JCF '97)
- Quasi-analytic approaches — Geman and Yor (MF '93) using Laplace transform; Rogers and Shi (JAP '95) using conditioning
Black-Scholes Setting

- **Security price:** \( S_t = S_0 \exp(\tilde{r}t + vB_t) \) where \( \tilde{r} = r - v^2/2 \)
- **Fixings:** \( m + 1 \) equally spaced fixings in \([0, T]\)
- **Price averages:** \( S_j := S_j \Delta, \quad j = 0, 1, \ldots, m \) where \( \Delta = T/m \)
- **Path-dependent state:** \( A_k \) (arithmetic) or \( G_k \) (geometric)

\[
A_k = \frac{1}{k+1} \sum_{j=0}^{k} S_j \quad \text{and} \quad G_k = \left( \prod_{j=0}^{k} S_j \right)^{1/(k+1)}
\]

- **Continuous analogues:**

\[
A_m \rightarrow A_T := \frac{1}{T} \int_{0}^{T} S_u \, du
\]

\[
G_m \rightarrow G_T := \exp \left( \frac{1}{T} \int_{0}^{T} \log S_u \, du \right)
\]
Pricing Formulas

• Fixed strike: \( C_0 \) (call) and \( P_0 \) (put)

\[
C_0 = e^{-rT} E[A_m - K]^+ \quad \text{and} \quad P_0 = e^{-rT} E[K - A_m]^+
\]

• Floating strike: \( \tilde{C}_0 \) (call) and \( \tilde{P}_0 \) (put)

\[
\tilde{C}_0 = e^{-rT} E[S_m - A_m]^+ \quad \text{and} \quad \tilde{P}_0 = e^{-rT} E[A_m - S_m]^+
\]

• Geometric average: replace \( A_m \) with \( G_m \)

\[
c_0 = S_0 e^{\tilde{\sigma}_m^2 / 2 - \kappa} N(\gamma_m + \tilde{\sigma}_m) - K N(\gamma_m)
\]

\[
\tilde{c}_0 = S_0 \left[ N\left( \frac{\kappa}{\tilde{\sigma}_m} \right) - e^{\tilde{\sigma}_m^2 / 2 - \kappa} N\left( \frac{\kappa}{\tilde{\sigma}_m} - \tilde{\sigma}_m \right) \right]
\]

where \( \gamma_m = \frac{\log(S_0/K) + \tilde{T}/2}{\tilde{\sigma}_m} \), \( \tilde{\sigma}_m = \sqrt{\frac{T(2m+1)}{6(m+1)}} \),

and \( \kappa = \left( r + \frac{v^2}{2} \right) \frac{T}{2} \)
Fixed Strike Asian Options

- $S_j = S_0 \exp(U_j)$, $j = 0, 1, \ldots, m$
  - $U_0 = 0$ and $U_j = X_1 + X_2 + \cdots + X_j$
  - $X_i$’s are independent $N(\mu, \sigma^2)$ random variables
  - $\mu = \tilde{r}\Delta$ and $\sigma = v\sqrt{\Delta}$

- $A_m = S_0(1 + e^{U_1} + \cdots + e^{U_m})/(m + 1)$

- At date $m - i$, the sum of the remaining $i + 1$ fixings is
  
  \[
  e^{U_{m-i}} + e^{U_{m-i+1}} + \cdots + e^{U_m} = e^{U_{m-i}} \times \\
  [1 + e^{X_{m-i+1}}(1 + e^{X_{m-i+2}} + \cdots + e^{X_{m-i+2} + \cdots + X_m})]
  \]

- $Y_i = \frac{1 + e^{X_{m-i+1}}(1 + e^{X_{m-i+2}} + \cdots + e^{X_{m-i+2} + \cdots + X_m})}{i + 1}$

- $Y_i \leq \frac{1 + iY_{i-1}e^X}{i + 1}$; $X \sim N(\mu, \sigma^2)$ independent of $Y_{i-1}$
Recursive Densities

- \( n(x) = e^{-x^2/2 \sqrt{2\pi}} \) and \( h_i = 1/(i + 1) \)

- Density of \( Y_1 \): for \( x > h_1 \)
  \[
  f_1(x) = \frac{1}{\sigma(x-h_1)} n \left( \frac{\log(x-h_1) - \log h_1 - \mu}{\sigma} \right)
  \]

- Density of \( Y_i \): for \( x > h_i \) (\( i = 2, \ldots, m \))
  \[
  f_i(x) = \frac{1}{\sigma(x-h_i)} \int_{h_{i-1}}^{\infty} \psi_i(x,y) f_{i-1}(y) \, dy
  \]
  where \( \psi_i(x,y) = n \left( \frac{\log(x-h_i) - \log(yh_i/h_{i-1}) - \mu}{\sigma} \right) \)

- Since \( A_m = S_0Y_m \),
  \[
  C_0 = e^{-rT} \int_{h_m}^{\infty} (S_0x - K)^+ f_m(x) \, dx
  \]
  \[
  = e^{-rT} \int_{K/S_0}^{\infty} (S_0x - K) f_m(x) \, dx \quad \text{if } K \geq h_m S_0
  \]
Recursive Numerical Integration

- **Truncate** the densities:
  - Value outside the “effective range” is negligible
  - Integrates to nearly 1 over the effective range

- Idea for determining good “truncation points”:
  - For arbitrary \( \varepsilon \), find the interval \((\underline{x}, \overline{x})\) such that
    \[ f_i(x) \leq \varepsilon \text{ whenever } x \notin (\underline{x}, \overline{x}) \]
  - Straightforward for \( f_1 \) but requires analysis of \( \psi_i(x, y) \)
    as a function of \( y \) (with \( x \) fixed) for \( f_2, \ldots, f_m \)
  - For fixed \( \alpha \) (small), express \( \varepsilon, \underline{x}, \overline{x} \) in terms of \( \alpha \) by
    requiring that
    \[ P(\underline{x} \leq Y_i \leq \overline{x}) \geq \alpha^i \]
  - Thus, \( f_i(x) \leq \varepsilon^*_i(\alpha) \) whenever \( x \notin (\underline{x}^*_i(\alpha), \overline{x}^*_i(\alpha)) \) and
    \[ \int_{\underline{x}^*_i(\alpha)}^{\overline{x}^*_i(\alpha)} f_i(x) \, dx \geq \alpha^i \] (refer to Proposition 1)
• Implementation of numerical integration:
  ○ Fix $\delta > 0$ (grid size) and $\overline{\alpha} > 0$ (coverage prob. of $f_m$)
  ○ $\alpha = \overline{\alpha}^{1/m}$, $\underline{k}_i = \lfloor x_i^*(\alpha)/\delta \rfloor$ and $\overline{k}_i = \lceil x_i^*(\alpha)/\delta \rceil$
  ○ Treat $f_i(y) = 0$ for $y \leq \underline{k}_i \delta$ and $y \geq \overline{k}_i \delta$
  ○ Approx. $f_i(y) = f_i(j\delta)$ for $(j - 1/2)\delta \leq y < (j + 1/2)\delta$
  ○ $f_i(x) = \delta \sum_{j = \underline{k}_i - 1 + 1}^{\overline{k}_i - 1} \psi_i(x, y_j) f_{i-1}(y_j)$ with $y_j = j\delta$
  ○ $C_0 = \delta e^{-rT} \sum_{j = \underline{k}_m + 1}^{\overline{k}_m - 1} (S_0x_j - K)^+ f_m(x_j)$ with $x_j = j\delta$.

• Comments:
  ○ Only compute $f_i(x)$ for $x = j\delta$, $j = \underline{k}_i + 1, \ldots, \overline{k}_i - 1$
  ○ After the $i$th iteration, store only the $\overline{k}_i - \underline{k}_i - 1$ values of $y_j$ and of $f_i(y_j)$ to be used in the next iteration
RNI vs. MC Simulation

• $S_0 = 100$, $r = 0.09$, $T = 1$, $m = 52$, $K = 100$

• For each $v = 0.05, 0.10, 0.30, 0.50$, generate 1000 repetitions of a 10000 simulation series from which a MC estimate is calculated (see Figure 1)

• Comments:
  • The precision of MC estimates is good for low volatility but deteriorates when $v$ (or $T$) increases (larger standard error)
  • For moderate to high volatility, more sampling repetitions need to be run for better precision at the expense of computational efficiency
  • RNI is superior to MC simulation
Lognormal Approximation

- Assume $\log Y_m \sim N(\alpha_m, \beta_m^2)$ with

$$\alpha_m = 2 \log E(Y_m) - \log E(Y_m^2)/2$$

$$\beta_m = \sqrt{\log E(Y_m^2) - 2 \log E(Y_m)}$$

where $E(Y_m)$ and $E(Y_m^2)$ are written in exact closed form

- Works well for low volatility, but the adequacy quickly degenerates as volatility increases (See Figure 2)

- True distribution of $\log Y_m$ is based on a 10000 MC simulation series and is standardized with $\alpha_m$ and $\beta_m$

- Lognormal approximation does not capture the tail behavior of the true distribution accurately
Floating Strike Asian Options

- Change of numeraire:
  - Equivalent to a change of measure
  - Work with \( S_t = S_0 \exp((r + v^2/2)t + v\tilde{B}_t) \)
  - Call: \( \tilde{C}_0 = e^{-rT}E[S_m - A_m]^+ = S_0\tilde{E}[1 - R_m]^+ \)
  - Put: \( \tilde{P}_0 = e^{-rT}E[A_m - S_m]^+ = S_0\tilde{E}[R_m - 1]^+ \)
  - \( R_k = A_k/S_k \): ratio at date \( k \)

- Since \( A_k = S_0Y_k \) and \( S_k = S_0e^{U_k} \),

\[
R_k = \frac{1 + e^{-X_k} + e^{-X_k-X_{k-1}} + \ldots + e^{-X_k-\cdots-X_1}}{k+1}
\]

- \( R_k \) has the same form as \( Y_k \)
  - \( -X \sim N(-\tilde{\mu}, \sigma^2) \) replaces \( X \sim N(\mu, \sigma^2) \)
  - \( \tilde{\mu} = (r + v^2/2)\Delta \) vs. \( \mu = (r - v^2/2)\Delta \)

- RNI procedure can be adapted to computing the density \( \tilde{f}_m \) of \( R_m \): use \(-r\) in place of \( r\)
• Relating floating strike option values to fixed strike option values
  
  ○ Call: \( \tilde{C}_0 = S_0 \int_{h_m}^{1} (1 - x) \tilde{f}_m(x) \, dx \)
  
  Related to the value of a fixed strike put option with spot = strike = 1, risk-free rate = \(-r\), volatility = \(v\), number of fixings = \(m\)

  ○ Put: \( \tilde{P}_0 = S_0 \int_{1}^{\infty} (x - 1) \tilde{f}_m(x) \, dx \)
  
  Related to the value of a fixed strike call option

• RNI compares favorably against other competing techniques, e.g., MC simulation (Figure 3): \(S_0 = 100, r = 0.10, m = 91 \ (T = 91/365, 182/365) \) or \(m = 121 \ (T = 364/365)\), \(v = 0.10, 0.20, 0.40\)
Continuous Asian Options

- Geometric average options:
  - As $m \to \infty$, $c_0 \to c_0^* := e^{-rT}E[G_T - K]^+$
  - $c_0^* = S_0e^{\tilde{\sigma}^2/2-\kappa}N(\gamma^* + \tilde{\sigma}^*) - KN(\gamma^*)$
  - $\tilde{\sigma}^* = \nu\sqrt{T/3}$ and $\gamma^* = (\log(S_0/K) + \bar{r}T/2)/\tilde{\sigma}^*$

- Rate of convergence is $O(m^{-1})$:
  - $c_0^* \doteq c_0 + \frac{\text{const.}}{m}$ for large $m$
  - const. independent of choice of $m$ (see Theorem 1)
  - Use Taylor series approximations to prove result

- Suggests a similar rate of convergence for arithmetic average options (see Table 3): $C_0^* \doteq C_0^* - \beta/m$
  - $C_0^* > 0$ and $\beta$ are parameters to be determined
  - $C_0^*$ is the price of a continuous Asian option
• Estimation of $C_0^* > 0$ and $\beta$ (see Table 4):
  ○ Richardson’s extrapolation
    Use the prices $C_1$ and $C_2$ based respectively on $m$ and $2m$ fixings
    Solve the equations $C_k = C_0^* - \beta/(km)$ ($k = 1, 2$):
    $$C_0^* = 2C_2 - C_1 \quad \text{and} \quad \beta = 2m(C_2 - C_1)$$
  ○ Least squares regression
    Fit a regression function of the form $C_0^* - \beta/m$ to the RNI estimates
    $C_0^*$ and $-\beta$ are respectively the LS estimates of the intercept and gradient for the regression line
    More computationally efficient than Richardson’s extrapolation

• Numerical comparisons (see Table 5)

• The approximate formula $C_0^* - \beta/m$ allows us to estimate the price of an option with any number of fixings
Discussion

- Through a sequence of recursive densities based on the univariate normal distribution, RNI allows us to price Asian options efficiently.
- RNI is applicable to both fixed strike and floating strike discrete Asian options.
- Accuracy of RNI price estimates are not sensitive to changes in volatility and/or time to maturity.
- Straightforward extensions take care of pricing prior to the averaging period, or into the averaging period.
- The relationship between option values and fixing frequency enables us to price Asian options with any number of fixings.
- RNI densities are themselves useful for the determination of a mixed density approximation for the true density of the discrete price average, thereby leading to new approximations for Asian option values.
Questions

- Obtain expressions for \( X_1, X_2, \ldots, X_m \) in terms of \( S_0, S_1, S_2, \ldots, S_m \) (and Brownian motion). Hence explain why the \( X_i \)'s are independent \( N(\mu, \sigma^2) \) random variables such that \( \mu = (r - v^2/2)\Delta \) and \( \sigma = v\sqrt{\Delta} \).

- Explain in your own words the main considerations in the implementation of the RNI algorithm. Also outline the steps in the algorithm.

- Show that in Richardson’s extrapolation, solving the equations \( C_k = C_k^* - \beta/(km) \) \((k = 1, 2)\) gives

  \[
  C_0^* = 2C_2 - C_1 \quad \text{and} \quad \beta = 2m(C_2 - C_1).
  \]

Could we have based \( C_1 \) and \( C_2 \) on fixing frequencies other than \( m \) and \( 2m \) respectively? Discuss.
FIGURE 1. Initial values of discrete fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $T = 1$, $m = 52$ and $K = 100$, through MC simulation (boxplots) and using RNI (solid lines). Summary statistics of 1000 simulation runs for each volatility are listed under the boxplots.
(a) $v = 0.10$ (low volatility): $\alpha_m = 0.04366$, $\beta_m = 0.05817$

(b) $v = 0.30$ (moderate volatility): $\alpha_m = 0.03002$, $\beta_m = 0.17510$

(c) $v = 0.50$ (high volatility): $\alpha_m = 0.00217$, $\beta_m = 0.29387$

FIGURE 2. The lognormal distribution as an approximation of the true distribution of the arithmetic average, for parameters $r = 0.09$, $T = 1$ and $m = 52$. (left) Quantile-plots against the standard normal distribution. (right) Comparison of lognormal densities (dotted lines) and RNI densities.
FIGURE 3. Initial values of discrete floating strike Asian call options with \( S_0 = 100 \) and \( r = 0.10 \), through MC simulation (boxplots), via the FSG algorithm and using RNI (dotted lines). The FSG results are based on Barraquand and Pudet (1996).
TABLE 3. Initial values of discrete fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $v = 0.30$ and $T = 1$ using RNI, for different values of $m$. Numbers in parentheses show increase in option price over the previous value. Continuous option values are obtained using the LS-6 scheme (see text and Table 4).

<table>
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<tr>
<th>$K$</th>
<th>$m$</th>
<th>26</th>
<th>52</th>
<th>104</th>
<th>208</th>
<th>416</th>
<th>832</th>
<th>$\infty^{\text{LS-6}}$</th>
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<td></td>
<td>14.9434</td>
<td>14.9629</td>
<td>(0.0195)</td>
<td>14.9728</td>
<td>(0.0099)</td>
<td>14.9779</td>
<td>(0.0051)</td>
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<td>95</td>
<td></td>
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<td>(0.0244)</td>
<td>11.6427</td>
<td>(0.0121)</td>
<td>11.6487</td>
<td>(0.0060)</td>
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<tr>
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<td>8.8006</td>
<td>(0.0264)</td>
<td>8.8143</td>
<td>(0.0137)</td>
<td>8.8210</td>
<td>(0.0067)</td>
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<tr>
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<td>(0.0268)</td>
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<td>6.5100</td>
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<td>(0.0063)</td>
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TABLE 4. Initial values $C_0^\beta$ and coefficient $\beta$ of continuous fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $v = 0.30$ and $T = 1$, estimated using Richardson’s extrapolation (RE-104 and RE-416) or least squares regression (LS-3 and LS-6).

<table>
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<tr>
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<th>RE-416</th>
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TABLE 5. Initial values of continuous fixed strike Asian call options with $S_0 = 100$ and $T = 1$, through PDE solution, via the R-LB and C-LB, and using RNI. The PDE solutions (headed by author initials) are due to Rogers and Shi (1995) and Zvan, Forsyth, and Vetzal (1997); the ZFV results are available for $r = 0.15$ only. The R-LB and C-LB results are based on Rogers and Shi (1995) and Chalasani, Jha, and Varikooty (1998) respectively.

<table>
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<th>$v$</th>
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<th>$r = 0.09$</th>
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