52. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that there will be
(a) at least 2 such accidents in the next month;
(b) at most 1 accident in the next month?

Explain your reasoning!

Let \( X \) = No. of airplane crashes of
commercial airlines per month

\[ X \sim \text{Poisson}(\lambda = 3.5) \] each flight has a
small prob. of crashing.

\[ P(X = k) = e^{-3.5} \frac{3.5^k}{k!}, \quad k = 0, 1, 2, \ldots \]

(a) \[ P(X \geq 2) = 1 - P(X = 0) - P(X = 1) \]
\[ = 1 - e^{-3.5} - 3.5 e^{-3.5} \]
\[ = 1 - 4.5 e^{-3.5} \]

(b) \[ P(X \leq 1) = P(X = 0) + P(X = 1) \]
\[ = e^{-3.5} + 3.5 e^{-3.5} = 4.5 e^{-3.5} \]
53. Approximately 80,000 marriages took place in the state of New York last year. Estimate the probability that for at least one of these couples
   (a) both partners were born on April 30;
   (b) both partners celebrated their birthday on the same day of the year.
   State your assumptions.

\[(a) \text{ Let } X = \text{ No. of both partners born on April 30.} \]

\[X \sim \text{Poisson } (\lambda) \text{ where } \lambda = 80000 \left(\frac{1}{365}\right)^2 \approx 0.6 \]

\[P(X \geq 1) = 1 - P(X = 0) \]

\[= 1 - e^{-0.6} \]

\[(b) \text{ Let } Y = \text{ No. of both partners born on the same day} \]

\[Y \sim \text{Poisson } (\lambda) \text{ where } \lambda = 80000 \left(\frac{1}{365}\right)^2 \times 365 \approx 219.18 \]

\[P(Y \geq 1) = 1 - e^{-219.18} \]
55. A certain typing agency employs 2 typists. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second. If your article is equally likely to be typed by either typist, approximate the probability that it will have no errors.

\[ P(\text{no error}) = 0.5 \cdot \frac{e^{-3} \cdot 3^0}{0!} + 0.5 \cdot \frac{e^{-4.2} \cdot 4.2^0}{0!} \]

\[ = 0.5e^{-3} + 0.5e^{-4.2} \]
60. The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter \( \lambda = 5 \). Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to \( \lambda = 3 \) for 75 percent of the population. For the other 25 percent of the population the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

\[
\begin{align*}
\text{Beneficial} & \quad 2 \text{ colds} \\
0.75 & \quad \lambda = 3 \\
\text{Not beneficial} & \quad \lambda = 5 \\
0.25 & \quad 2 \text{ colds}
\end{align*}
\]

Bayes' theorem:

\[
P (\text{beneficial} \mid 2 \text{ colds}) = \frac{0.75 \left[ e^{-3} \frac{3^2}{2!} \right]}{0.75 \left[ e^{-3} \frac{3^2}{2!} \right] + 0.25 \left[ e^{-5} \frac{5^2}{2!} \right]}
\]
The suicide rate in a certain state is 1 suicide per 100,000 inhabitants per month.

(a) Find the probability that in a city of 400,000 inhabitants within this state, there will be 8 or more suicides in a given month.

(b) What is the probability that there will be at least 2 months during the year that will have 8 or more suicides?

(c) Counting the present month as month number 1, what is the probability that the first month to have 8 or more suicides will be month number \( i \), \( i \geq 1 \)?

What assumptions are you making?

(a) Let \( X = \) No. of suicides in a given month.

\[
X \sim \text{Poisson} (\lambda), \quad \lambda = 400000 \left( \frac{1}{100000} \right) = 4
\]

\[
P(X = k) = \frac{e^{-4} \cdot 4^k}{k!}, \quad k = 0, 1, 2, \ldots
\]

\[
P(X \geq 8) = 1 - P(X \leq 7) = 1 - \sum_{k=0}^{7} \frac{e^{-4} \cdot 4^k}{k!}
\]

(b) Let \( Y = \) No. of months that will have 8 or more suicides.

\( Y \sim \text{Binomial} \left( n = 12, \quad p = P(X \geq 8) \right) \)

\[
P(Y = i) = \binom{12}{i} p^i (1-p)^{12-i}, \quad i = 0, 1, \ldots, 12
\]
\[ P(y \geq 2) = 1 - P(y = 0) - P(y = 1) \]
\[ = 1 - (1-p)^2 - 12 p (1-p)^2. \]

(c) Let \( z \) = No. of months until a month has 8 or more suicides.

\[ z \sim \text{Geometric} \left( p \equiv P(X \geq 8) \right). \]

\[ P(z = j) = (1-p)^{j-1} \cdot p, \quad j = 1, 2, \ldots \]

\[ P(z = i) = (1-p)^{i-1} \cdot p. \]
65. Each of 500 soldiers in an army company independently has a certain disease with probability $1/10^3$. This disease will show up in a blood test, and to facilitate matters blood samples from all 500 are pooled and tested.

(a) What is the (approximate) probability that the blood test will be positive (and so at least one person has the disease)?

Suppose now that the blood test yields a positive result.

(b) What is the probability, under this circumstance, that more than one person has the disease?

One of the 500 people is Jones, who knows that he has the disease.

(c) What does Jones think is the probability that more than one person has the disease?

As the pooled test was positive, the authorities have decided to test each individual separately. The first $i - 1$ of these tests were negative, and the $i$th one—which was on Jones—was positive.

(d) Given the preceding, as a function of $i$, what is the probability that any of the remaining people have the disease?

(e) Let $X =$ No. of soldiers with the disease.

$$X \sim \text{Poisson}(\lambda), \quad \lambda = 500 \left(\frac{1}{1000}\right) = \frac{1}{2}$$

$$P(X = k) = e^{-\frac{1}{2}} \cdot \left(\frac{1}{2}\right)^k \cdot \frac{1}{k!}, \quad k = 0, 1, 2, \ldots$$

$$P(X > 1) = 1 - P(X = 0) = 1 - e^{-\frac{1}{2}}.$$
65. (cont.)

(b) \( P(X > 1 \mid X > 1) = \frac{P(X > 1 \cap X > 1)}{P(X > 1)} \)

= \frac{P(X > 2)}{P(X > 1)}

= \frac{1 - e^{-\frac{1}{2}} - \frac{1}{2} e^{-\frac{1}{2}}}{1 - e^{-\frac{1}{2}}} \quad \text{with } X > 2 \text{ out of 500}

(c) \( P(X > 1 \mid \text{Jones has}) = \frac{P(X > 1 \cap \text{Jones has})}{P(\text{Jones has})} \)

= \frac{P(X > 1)}{P(\text{Jones has})}

= \frac{1 - e^{-\frac{1}{2}}}{1} = 1 - e^{-\frac{1}{2}} \quad \text{out of 499}
(d) Let \( R = \) no. of remaining people with the disease.

\[
P(R \geq 1 \mid \text{NN \ldots NP}) \quad \text{negative} \quad \text{positive} \quad \text{Jones}
\]

\[
i-1 \quad \text{independent}
\]

\[
= 1 - e^{-\frac{500 - i}{1000}}
\]
74. An interviewer is given a list of potential people she can interview. If the interviewer needs to interview 5 people and if each person (independently) agrees to be interviewed with probability \( \frac{2}{3} \), what is the probability that her list of potential people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people? For part (b) what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

(a) \( \binom{5}{5} \left( \frac{2}{3} \right)^5 \left( \frac{1}{3} \right)^0 \)

(b) \( \binom{8}{5} \left( \frac{2}{3} \right)^5 \left( \frac{1}{3} \right)^3 + \binom{8}{6} \left( \frac{2}{3} \right)^6 \left( \frac{1}{3} \right)^2 + \binom{8}{7} \left( \frac{2}{3} \right)^7 \left( \frac{1}{3} \right)^1 + \left( \frac{2}{3} \right)^8 \)

(c) 

\[ \text{Fixed} \]

Negative binomial: \( \binom{5}{4} \left( \frac{2}{3} \right)^5 \left( \frac{1}{3} \right) \)

(d) 

\[ \text{Fixed} \]

Negative binomial: \( \binom{6}{4} \left( \frac{2}{3} \right)^5 \left( \frac{1}{3} \right)^2 \)
An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly \( n \) selections?

Let \( X \) = No. of selections made

\[
X \sim \text{Geometric} (p) \quad \text{where} \quad p = \frac{\binom{4}{2} \binom{4}{2}}{\binom{8}{4}}
\]

\[
P(X = k) = (1-p)^{k-1} \cdot p, \quad k = 1, 2, \ldots
\]

\[
P(X = n) = (1-p)^{n-1} \cdot p.
\]
6. For a nonnegative integer-valued random variable $N$, show that

$$E[N] = \sum_{i=1}^{\infty} P(N \geq i)$$

**Hint:** $\sum_{i=1}^{\infty} P(N \geq i) = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(N = k)$. Now interchange the order of summation.

$$E(N) = \sum_{i=1}^{\infty} i \cdot P(N = i)$$

$$= 1 \cdot P(N = 1) + 2 \cdot P(N = 2) + 3 \cdot P(N = 3) + \ldots$$

$$= P(N = 1) + P(N = 2) + P(N = 3) + \ldots$$

$$= \sum_{i=1}^{\infty} P(N \geq i)$$
9. Let $X$ be a random variable having expected value $\mu$ and variance $\sigma^2$. Find the expected value and variance of

$$Y = \frac{X - \mu}{\sigma}$$

$$Y = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} \mathbb{E}(X) - \frac{\mu}{\sigma}$$

$$= \frac{1}{\sigma} \cdot \mu - \frac{\mu}{\sigma}$$

$$= 0$$

$$\text{Var}(Y) = \text{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$$

$$= \left(\frac{1}{\sigma}\right)^2 \text{Var}(X)$$

$$= \frac{1}{\sigma^2} \cdot \sigma^2$$

$$= 1.$$
10. Let $X$ be a binomial random variable with parameters $n$ and $p$. Show that

$$E \left[ \frac{1}{X + 1} \right] = \frac{1 - (1 - p)^{n+1}}{(n + 1) p}$$

$$E \left( \frac{1}{X+1} \right) = \sum_{i=0}^{n} \frac{1}{i+1} \frac{n!}{(n-i)! \cdot i!} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{n!}{(n-i)! \cdot (i+1)!} p^i (1-p)^{n-i}$$

$$= \frac{1}{(n+1)p} \sum_{i=0}^{n} \frac{(n+1)!}{(n-i)! \cdot (i+1)!} p^i (1-p)^{n-i}$$

$$= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \frac{(n+1)!}{(n-(j-1))! \cdot j!} p^j (1-p)^{n-j+1}$$

(let $j = i + 1$)

$$= \frac{1}{(n+1)p} \left[ 1 - \binom{n+1}{0} p^0 (1-p)^{n+1-0} \right]$$

$$= \frac{1}{(n+1)p} \left[ 1 - (1-p)^{n+1} \right].$$
19. If $X$ is a Poisson random variable with parameter $\lambda$, show that

$$E[X^n] = \lambda E[(X + 1)^{n-1}]$$

Now use this result to compute $E[X^3]$.

$$E(X^n) = \sum_{i=0}^{\infty} i^n \frac{e^{-\lambda} \lambda^i}{i!}$$

$$= \sum_{i=1}^{\infty} i^{n-1} \frac{e^{-\lambda} \lambda^i}{(i-1)!}$$

$$= \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{e^{-\lambda} \lambda^{j+1}}{j!}$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{e^{-\lambda} \lambda^j}{j!}$$

$$= \lambda E[(X+1)^{n-1}]$$

\[ E(x) = \lambda \ E[(x+1)^0] = \lambda \]

\[ E(x^2) = \lambda \ E[(x+1)^1] = \lambda \ [E(x) + 1] = \lambda (\lambda + 1) = \lambda^2 + \lambda \]

\[ E(x^3) = \lambda \ E[(x+1)^2] \]

\[ = \lambda \ E[x^2 + 2x + 1] \]

\[ = \lambda [E(x^2) + 2E(x) + 1] \]

\[ = \lambda [\lambda^2 + \lambda + 2\lambda + 1] \]

\[ = \lambda (\lambda^2 + 3\lambda + 1) \]
21. From a set of $n$ randomly chosen people let $E_{ij}$ denote the event that persons $i$ and $j$ have the same birthday. Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday. Find

(a) $P(E_{3,4}|E_{1,2})$;
(b) $P(E_{1,3}|E_{1,2})$;
(c) $P(E_{2,3}|E_{1,2} \cap E_{1,3})$.

What can you conclude from the above about the independence of the $\binom{n}{2}$ events $E_{ij}$?

(a) $P\left( E_{3,4} \left| E_{1,2} \right. \right) = P\left( E_{3,4} \right) = \frac{1}{365}$

(b) $P\left( E_{1,3} \left| E_{1,2} \right. \right) = P\left( E_{1,3} \right) = \frac{1}{365}$

(c) $P\left( E_{2,3} \left| E_{1,2} \cap E_{1,3} \right. \right) = 1$.

$E_{ij}$'s are pairwise independent.
27. If $X$ is a geometric random variable, show analytically that

$$P(X = n + k | X > n) = P(X = k)$$

Give a verbal argument using the interpretation of a geometric random variable as to why the preceding equation is true.

\[
P(X = n + k | X > n) \\
= \frac{P(X = n + k \cap X > n)}{P(X > n)} \\
= \frac{P(X = n + k)}{P(X > n)} \\
= \frac{P(1-p)^{n+k-1}}{(1-p)^n} \\
= p(1-p)^{k-1}
\]

If the first $n$ trials are all failures, then getting a success at $(n+k)$th trial is the same as a fresh start and getting a success at the $k$th trial.
28. Let $X$ be a negative binomial random variable with parameters $r$ and $p$, and let $Y$ be a binomial random variable with parameters $n$ and $p$. Show that

$$P(X > n) = P(Y < r)$$

*Hint:* One could either attempt an analytical proof of the preceding, which is equivalent to proving the identity

$$\sum_{i=n+1}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{i-r} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i}$$

or one could attempt a proof that uses the probabilistic interpretation of these random variables. That is, in the latter case start by considering a sequence of independent trials having a common success probability $p$. Then try to express the events $\{X > n\}$ and $\{Y < r\}$ in terms of the outcomes of this sequence.

$\{X > n\}$ is the event that the no. of trials needed to get $r$ successes is more than $n$.

$\{Y < r\}$ is the event that the no. of successes out of $n$ trials is less than $r$.

Less than $r$ successes

\[\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
n^\text{th trial}
\end{array}\]

Note that $\{X > n\} = \{Y < r\}$, thus $P(X > n) = P(Y < r)$. 