Unbiased tests

When a UMP test does not exist, we may use the same approach used in estimation problems, i.e., imposing a reasonable restriction on the tests. One such restriction is unbiasedness. A UMP test $T$ of size $\alpha$ has the property that

$$\beta_T(P) \leq \alpha, \quad P \in \mathcal{P}_0 \quad \text{and} \quad \beta_T(P) \geq \alpha, \quad P \in \mathcal{P}_1, \quad (1)$$

since $T$ is at least as good as the silly test $T \equiv \alpha$. This leads to the following definition.

**Definition 6.3**

Let $\alpha$ be a given level of significance. A test $T$ for $H_0 : P \in \mathcal{P}_0$ versus $H_1 : P \in \mathcal{P}_1$ is said to be unbiased of level $\alpha$ if and only if (1) holds. A test of size $\alpha$ is called a *uniformly most powerful unbiased* (UMPU) test if and only if it is UMP within the class of unbiased tests of level $\alpha$.

The consideration of UMPU tests can be confined to the tests based on a sufficient statistics. Why?
UMP tests in exponential families

Consider the following hypotheses:

\[ H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1, \]

where \( \theta = \theta(P) \) is a functional from \( \mathcal{P} \) onto \( \Theta \) and \( \Theta_0 \) and \( \Theta_1 \) are disjoint and \( \Theta_0 \cup \Theta_1 = \Theta \). (\( \mathcal{P}_j = \{P : \theta \in \Theta_j\}, j = 0, 1.\))

**Definition 6.4 (Similarity)**

Consider the hypotheses \( H_0 : \theta \in \Theta_0 \) vs \( H_1 : \theta \in \Theta_1 \). Let \( \alpha \) be a given level of significance and let \( \bar{\Theta}_{01} \) be the common boundary of \( \Theta_0 \) and \( \Theta_1 \), i.e., the set of points \( \theta \) that are points or limit points of both \( \Theta_0 \) and \( \Theta_1 \).

A test \( T \) is similar on \( \bar{\Theta}_{01} \) if and only if \( \beta_T(P) = \alpha \) for all \( \theta \in \bar{\Theta}_{01} \).

**Remark**

It is more convenient to work with similarity than to work with unbiasedness for testing \( H_0 : \theta \in \Theta_0 \) vs \( H_1 : \theta \in \Theta_1 \).
Continuity of the power function

For a given test $T$, the power function $\beta_T(P)$ is said to be continuous in $\theta$ if and only if for any $\{\theta_j : j = 0, 1, 2, \ldots\} \subset \Theta$, $\theta_j \to \theta_0$ implies $\beta_T(P_j) \to \beta_T(P_0)$, where $P_j \in \mathcal{P}$ satisfying $\theta(P_j) = \theta_j, j = 0, 1, \ldots$. If $\beta_T$ is a function of $\theta$, then this continuity property is simply the continuity of $\beta_T(\theta)$.

**Lemma 6.5**

Consider hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$. Suppose that, for every $T$, $\beta_T(P)$ is continuous in $\theta$. If $T_*$ is uniformly most powerful among all similar tests and has size $\alpha$, then $T_*$ is a UMPU test.

**Proof**

Under the continuity assumption on $\beta_T$, the class of similar tests contains the class of unbiased tests. Since $T_*$ is uniformly at least as powerful as the test $T \equiv \alpha$, $T_*$ is unbiased. Hence, $T_*$ is a UMPU test.
Neyman structure

Let $U(X)$ be a sufficient statistic for $P \in \bar{P} = \{P : \theta \in \bar{\Theta}_{01}\}$ and let $\bar{P}_U$ be the family of distributions of $U$ as $P$ ranges over $\bar{P}$. A test is said to have Neyman structure w.r.t. $U$ if

$$E[T(X)|U] = \alpha \quad \text{a.s. } \bar{P}_U,$$

Clearly, if $T$ has Neyman structure, then

$$E[T(X)] = E\{E[T(X)|U]\} = \alpha \quad P \in \bar{P},$$

i.e., $T$ is similar on $\bar{\Theta}_{01}$. If all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. $U$, then working with tests having Neyman structure is the same as working with tests similar on $\bar{\Theta}_{01}$.

Lemma 6.6

Let $U(X)$ be a sufficient statistic for $P \in \bar{P}$. A necessary and sufficient condition for all tests similar on $\bar{\Theta}_{01}$ to have Neyman structure w.r.t. $U$ is that $U$ is boundedly complete for $P \in \bar{P}$. 
Proof

(i) Suppose first that \( U \) is boundedly complete for \( P \in \bar{P} \).
Let \( T(X) \) be a test similar on \( \bar{\Theta}_{01} \).
Then \( E[T(X) - \alpha] = 0 \) for all \( P \in \bar{P} \).
From the boundedness of \( T(X) \), \( E[T(X)|U] \) is bounded.
Since \( E\{E[T(X)|U] - \alpha\} = E[T(X) - \alpha] = 0 \) for all \( P \in \bar{P} \) and \( U \)
is boundedly complete, \( E[T(X)|U] = \alpha \) a.s. \( \bar{P}_U \), i.e., \( T \) has
Neyman structure.

(ii) Suppose now that all tests similar on \( \bar{\Theta}_{01} \) have Neyman
structure w.r.t. \( U \).
Suppose also that \( U \) is not boundedly complete for \( P \in \bar{P} \).
Then there is a function \( h \) such that \( |h(u)| \leq C, E[h(U)] = 0 \) for
all \( P \in \bar{P} \), and \( h(U) \neq 0 \) with positive probability for some \( P \in \bar{P} \).
Let \( T(X) = \alpha + ch(U) \), where \( c = \min\{\alpha, 1 - \alpha\}/C \).
Then \( T \) is a test similar on \( \bar{\Theta}_{01} \) but \( T \) does not have Neyman
structure w.r.t. \( U \) (because \( h(U) \neq 0 \)).
Thus, \( U \) must be boundedly complete for \( P \in \bar{P} \).
This proves the result.
Theorem 6.4 (UMPU tests in exponential families)

Suppose that $X$ has the following p.d.f. w.r.t. a $\sigma$-finite measure:

$$f_{\theta, \varphi}(x) = \exp \{\theta Y(x) + \varphi^T U(x) - \zeta(\theta, \varphi)\},$$

where $\theta$ is a real-valued parameter, $\varphi$ is a vector-valued parameter, and $Y$ (real-valued) and $U$ (vector-valued) are statistics.

(i) For testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, a UMPU test of size $\alpha$ is

$$T^*_*(Y, U) = \begin{cases} 1 & Y > c(U) \\ \gamma(U) & Y = c(U) \\ 0 & Y < c(U), \end{cases}$$

where $c(u)$ and $\gamma(u)$ are Borel functions determined by

$$E_{\theta_0}[T^*_*(Y, U)|U = u] = \alpha \text{ for every } u$$

and $E_{\theta_0}$ is the expectation w.r.t. $f_{\theta_0, \varphi}$.

(ii) For testing $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ versus $H_1 : \theta_1 < \theta < \theta_2$, a UMPU test of size $\alpha$ is

$$T^*_*(Y, U) = \begin{cases} 1 & c_1(U) < Y < c_2(U) \\ \gamma_i(U) & Y = c_i(U), \ i = 1, 2, \\ 0 & Y < c_1(U) \text{ or } Y > c_2(U), \end{cases}$$
where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha \quad \text{for every } u.
\]

(iii) For testing \( H_0 : \theta_1 \leq \theta \leq \theta_2 \) versus \( H_1 : \theta < \theta_1 \) or \( \theta > \theta_2 \), a UMPU test of size \( \alpha \) is

\[
T_*(Y, U) = \begin{cases} 
1 & \text{if } Y < c_1(U) \text{ or } Y > c_2(U) \\
\gamma_i(U) & \text{if } Y = c_i(U), \ i = 1, 2, \\
0 & \text{if } c_1(U) < Y < c_2(U),
\end{cases}
\]

where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha \quad \text{for every } u.
\]

(iv) For testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \neq \theta_0 \), a UMPU test of size \( \alpha \) is given by \( T_*(Y, U) \) in (iii), where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_0}[T_*(Y, U)|U = u] = \alpha \quad \text{for every } u
\]

and

\[
E_{\theta_0}[T_*(Y, U)Y|U = u] = \alpha E_{\theta_0}(Y|U = u) \quad \text{for every } u.
\]
Proof

By sufficiency, we only need to consider tests that are functions of \((Y, U)\).

It follows from Theorem 2.1(i) that the p.d.f. of \((Y, U)\) (w.r.t. a \(\sigma\)-finite measure) is in a natural exponential family of the form \(\exp \{\theta y + \varphi^T u - \zeta(\theta, \varphi)\}\) and, given \(U = u\), the p.d.f. of the conditional distribution of \(Y\) (w.r.t. a \(\sigma\)-finite measure \(\nu_u\)) is in a natural exponential family of the form \(\exp \{\theta y - \zeta_u(\theta)\}\).

Hypotheses in (i)-(iv) are of the form \(H_0 : \theta \in \Theta_0\) vs \(H_1 : \theta \in \Theta_1\) with \(\bar{\Theta}_{01} = \{(\theta, \varphi) : \theta = \theta_0\}\) or \(\{(\theta, \varphi) : \theta = \theta_i, i = 1, 2\}\).

In case (i) or (iv), \(U\) is sufficient and complete for \(P \in \bar{P}\) and, hence, Lemma 6.6 applies.

In case (ii) or (iii), applying Lemma 6.6 to each \(\{(\theta, \varphi) : \theta = \theta_i\}\) also shows that working with tests having Neyman structure is the same as working with tests similar on \(\bar{\Theta}_{01}\).

By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.
Thus, for (i), it suffices to show $T_\ast$ is UMP among all tests $T$ satisfying
\begin{equation}
E_{\theta_0}[T(Y, U)|U = u] = \alpha \quad \text{for every } u
\end{equation}
and for part (ii) or (iii)), it suffices show $T_\ast$ is UMP among all tests $T$ satisfying
\begin{equation}
E_{\theta_1}[T(Y, U)|U = u] = E_{\theta_2}[T(Y, U)|U = u] = \alpha \quad \text{for every } u.
\end{equation}
For (iv), any unbiased $T$ should satisfy (2) and
\begin{equation}
\frac{\partial}{\partial \theta} E_{\theta, \varphi}[T(Y, U)] = 0, \quad \theta \in \bar{\Theta}_{01}.
\end{equation}
One can show (exercise) that (3) is equivalent to
\begin{equation}
E_{\theta, \varphi}[T(Y, U)Y - \alpha Y] = 0, \quad \theta \in \bar{\Theta}_{01}.
\end{equation}
Using the argument in the proof of Lemma 6.6, one can show (exercise) that (4) is equivalent to
\begin{equation}
E_{\theta_0}[T(Y, U)Y|U = u] = \alpha E_{\theta_0}(Y|U = u) \quad \text{for every } u.
\end{equation}
Hence, for (iv), it suffices to show $T_\ast$ is UMP among all tests $T$ satisfying (2) and (5).
Note that the power function of any test \( T(Y, U) \) is
\[
\beta_T(\theta, \varphi) = \int \left[ \int T(y, u) dP_{Y|U=u}(y) \right] dP_U(u).
\]
Thus, it suffices to show that for every fixed \( u \) and \( \theta \in \Theta_1 \), \( T_* \) maximizes
\[
\int T(y, u) dP_{Y|U=u}(y)
\]
over all \( T \) subject to the given side conditions.
Since \( P_{Y|U=u} \) is in a one-parameter exponential family, the results in (i) and (ii) follow from Corollary 6.1 and Theorem 6.3, respectively. The result in (iii) follows from Theorem 6.3(ii) by considering \( 1 - T_* \).

To prove the result in (iv), it suffices to show that if \( Y \) has the p.d.f. given by \( \exp \{ \theta y - \zeta_u(\theta) \} \) and if \( u \) is treated as a constant in (2) and (5), \( T_* \) in (iii) with a fixed \( u \) is UMP subject to conditions (2) and (5). We now omit \( u \) in the following proof for (iv), which is very similar to the proof of Theorem 6.3.
First, \((\alpha, \alpha E_{\theta_0}(Y))\) is an interior point of the set of points \((E_{\theta_0}[T(Y)], E_{\theta_0}[T(Y)Y])\) as \(T\) ranges over all tests of the form \(T(Y)\). By Lemma 6.2 and Proposition 6.1, for testing \(\theta = \theta_0\) versus \(\theta = \theta_1\), the UMP test is equal to 1 when

\[
(k_1 + k_2 y)e^{\theta_0 y} < C(\theta_0, \theta_1)e^{\theta_1 y},
\]

where \(k_i\)'s and \(C(\theta_0, \theta_1)\) are constants. This inequality is equivalent to

\[
a_1 + a_2 y < e^{by}
\]

for some constants \(a_1\), \(a_2\), and \(b\). This region is either one-sided or the outside of an interval. By Theorem 6.2(ii), a one-sided test has a strictly monotone power function and therefore cannot satisfy (5). Thus, this test must have the form of \(T_*\) in (iii). Since \(T_*\) in (iii) does not depend on \(\theta_1\), by Lemma 6.1, it is UMP over all tests satisfying (2) and (5); in particular, the test \(\equiv \alpha\). Thus, \(T_*\) is UMPU.

Finally, it can be shown that all the \(c\)- and \(\gamma\)-functions in (i)-(iv) are Borel functions of \(u\) (see Lehmann (1986, p. 149)).
Example 6.11
A problem arising in many different contexts is the comparison of two treatments. If the observations are integer-valued, the problem often reduces to testing the equality of two Poisson distributions (e.g., a comparison of the radioactivity of two substances or the car accident rate in two cities) or two binomial distributions (when the observation is the number of successes in a sequence of trials for each treatment).

Consider first the Poisson problem in which $X_1$ and $X_2$ are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively. The p.d.f. of $X = (X_1, X_2)$ is

$$
\left[ e^{-(\lambda_1+\lambda_2)} / x_1! x_2! \right] \exp \left\{ x_2 \log(\lambda_2/\lambda_1) + (x_1 + x_2) \log \lambda_1 \right\}
$$

w.r.t. the counting measure on

$$
\{(i,j) : i = 0, 1, 2, ..., j = 0, 1, 2, ...\}.
$$
Example 6.11 (continued)

Let $\theta = \log(\lambda_2/\lambda_1)$. Then hypotheses such as $\lambda_1 = \lambda_2$ and $\lambda_1 \geq \lambda_2$ are equivalent to $\theta = 0$ and $\theta \leq 0$, respectively.

The p.d.f. of $X$ is in a multiparameter exponential family with $\varphi = \log \lambda_1$, $Y = X_2$, and $U = X_1 + X_2$.

Thus, Theorem 6.4 applies.

To obtain various tests in Theorem 6.4, it is enough to derive the conditional distribution of $Y = X_2$ given $U = X_1 + X_2 = u$.

Using the fact that $X_1 + X_2$ has the Poisson distribution $P(\lambda_1 + \lambda_2)$, one can show that

$$P(Y = y|U = u) = \binom{u}{y} p^y (1-p)^{u-y} I_{\{0,1,\ldots,u\}}(y), \quad u = 0, 1, 2, \ldots,$$

where $p = \lambda_2/(\lambda_1 + \lambda_2) = e^\theta/(1 + e^\theta)$.

This is the binomial distribution $Bi(p, u)$.

On the boundary set $\bar{\Theta}_{01}$, $\theta = \theta_j$ (a known value) and the distribution $P_{Y|U=u}$ is known.
Example 6.11 (continued)

Consider next the binomial problem in which $X_j, j = 1, 2$, are independently distributed as the binomial distributions $Bi(p_j, n_j)$, $j = 1, 2$, respectively, where $n_j$’s are known but $p_j$’s are unknown. The p.d.f. of $X = (X_1, X_2)$ is

$$
\binom{n_1}{x_1} \binom{n_2}{x_2} (1 - p_1)^{n_1} (1 - p_2)^{n_2} \exp \left\{ x_2 \log \frac{p_2(1-p_1)}{p_1(1-p_2)} + (x_1 + x_2) \log \frac{p_1}{(1-p_1)} \right\}
$$

w.r.t. the counting measure on

$$\{(i,j) : i = 0, 1, ..., n_1, j = 0, 1, ..., n_2\}.$$

This p.d.f. is in a multiparameter exponential family with

$$\theta = \log \frac{p_2(1-p_1)}{p_1(1-p_2)}, \ Y = X_2, \ \text{and} \ U = X_1 + X_2.$$

Thus, Theorem 6.4 applies.

Note that hypotheses such as $p_1 = p_2$ and $p_1 \geq p_2$ are equivalent to $\theta = 0$ and $\theta \leq 0$, respectively.
Example 6.11 (continued)

Using the joint distribution of \((X_1, X_2)\), one can show (exercise) that

\[
P(Y = y | U = u) = K_u(\theta) \binom{n_1}{u - y} \binom{n_2}{y} e^{\theta y} I_A(y), \quad u = 0, 1, \ldots, n_1 + n_2,
\]

where

\[
A = \{y : y = 0, 1, \ldots, \min\{u, n_2\}, u - y \leq n_1\}
\]

and

\[
K_u(\theta) = \left[ \sum_{y \in A} \binom{n_1}{u - y} \binom{n_2}{y} e^{\theta y} \right]^{-1}.
\]

If \(\theta = 0\), this distribution reduces to a known distribution: the hypergeometric distribution \(HG(u, n_2, n_1)\) (Table 1.1, page 18).