Monotone likelihood ratio
A simple hypothesis involves only one population. If a hypothesis is not simple, it is called composite. UMP tests for a composite $H_1$ exist in Example 6.2.

We now extend this result to a class of parametric problems in which the likelihood functions have a special property.

Definition 6.2
Suppose that the distribution of $X$ is in $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, a parametric family indexed by a real-valued $\theta$, and that $\mathcal{P}$ is dominated by a $\sigma$-finite measure $\nu$. Let $f_\theta = dP_\theta/d\nu$.

The family $\mathcal{P}$ is said to have monotone likelihood ratio in $Y(X)$ (a real-valued statistic) if and only if, for any $\theta_1 < \theta_2$, $f_{\theta_2}(x)/f_{\theta_1}(x)$ is a nondecreasing function of $Y(x)$ for values $x$ at which at least one of $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ is positive.
Lemma 6.3
Suppose that the distribution of $X$ is in a parametric family $\mathcal{P}$ indexed by a real-valued $\theta$ and that $\mathcal{P}$ has monotone likelihood ratio in $Y(X)$. If $\psi$ is a nondecreasing function of $Y$, then $g(\theta) = E[\psi(Y)]$ is a nondecreasing function of $\theta$.

Proof:
Let $\theta_1 < \theta_2$, $A = \{x : f_{\theta_1}(x) > f_{\theta_2}(x)\}$, $B = \{x : f_{\theta_1}(x) < f_{\theta_2}(x)\}$, $a = \sup_{x \in A} \psi(Y(x))$, $b = \inf_{x \in B} \psi(Y(x))$. Then $b \geq a$, and

$$g(\theta_2) - g(\theta_1) = \int \psi(Y(x))(f_{\theta_2} - f_{\theta_1})(x)d\nu$$

$$\geq a \int_A (f_{\theta_2} - f_{\theta_1})(x)d\nu + b \int_B (f_{\theta_2} - f_{\theta_1})(x)d\nu$$

$$= (b - a) \int_B (f_{\theta_2} - f_{\theta_1})(x)d\nu \geq 0.$$

Take $\psi(y) = I_{(t, \infty)}(y)$. Then $g(\theta) = P(Y > t) = 1 - F_Y(t)$ is nondecreasing in $\theta$. 
Example 6.3
Let \( \theta \) be real-valued and \( \eta(\theta) \) be a nondecreasing function of \( \theta \). Then the one-parameter exponential family with

\[
f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x)
\]

has monotone likelihood ratio in \( Y(X) \).

Example 6.4
Let \( X_1, ..., X_n \) be i.i.d. from the uniform distribution on \((0, \theta)\), where \( \theta > 0 \). The Lebesgue p.d.f. of \( X = (X_1, ..., X_n) \) is\( f_\theta(x) = \theta^{-n}I_{(0,\theta)}(x(n)) \), where \( x(n) \) is the value of the largest order statistic \( X(n) \). For \( \theta_1 < \theta_2 \),

\[
\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{\theta_1^n}{\theta_2^n} \frac{I_{(0,\theta_2)}(x(n))}{I_{(0,\theta_1)}(x(n))},
\]

which is a nondecreasing function of \( x(n) \) for \( x \)'s at which at least one of \( f_{\theta_1}(x) \) and \( f_{\theta_2}(x) \) is positive, i.e., \( x(n) < \theta_2 \). Hence the family of distributions of \( X \) has monotone likelihood ratio in \( X(n) \).
Example 6.5

The following families have monotone likelihood ratio:

- the double exponential distribution family \( \{DE(\theta, c)\} \) with a known \( c \);
- the exponential distribution family \( \{E(\theta, c)\} \) with a known \( c \);
- the logistic distribution family \( \{LG(\theta, c)\} \) with a known \( c \);
- the hypergeometric distribution family \( \{HG(r, \theta, N - \theta)\} \) with known \( r \) and \( N \) (Table 1.1, page 18).

An example of a family that does not have monotone likelihood ratio is the Cauchy distribution family \( \{C(\theta, c)\} \) with a known \( c \).

Testing one sided hypotheses

Hypotheses of the form \( H_0 : \theta \leq \theta_0 \) (or \( H_0 : \theta \geq \theta_0 \)) versus \( H_1 : \theta > \theta_0 \) (or \( H_1 : \theta < \theta_0 \)) are called one-sided hypotheses for any fixed constant \( \theta_0 \).
Theorem 6.2
Suppose that $X$ has a distribution in $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ ($\Theta \subset \mathbb{R}$) that has monotone likelihood ratio in $Y(X)$. Consider the problem of testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0$ is a given constant.

(i) There exists a UMP test of size $\alpha$, which is given by

$$T^*_*(X) = \begin{cases} 1 & Y(X) > c \\ \gamma & Y(X) = c \\ 0 & Y(X) < c, \end{cases}$$

where $c$ and $\gamma$ are determined by $\beta_{T^*_*}(\theta_0) = \alpha$, and $\beta_T(\theta) = E[T(X)]$ is the power function of a test $T$.

(ii) $\beta_{T^*_*}(\theta)$ is strictly increasing for all $\theta$’s for which $0 < \beta_{T^*_*}(\theta) < 1$.

(iii) For any $\theta < \theta_0$, $T_*$ minimizes $\beta_T(\theta)$ (the type I error probability of $T$) among all tests $T$ satisfying $\beta_T(\theta_0) = \alpha$. 
(iv) Assume that $P_\theta(f_\theta(X) = cf_{\theta_0}(X)) = 0$ for any $\theta > \theta_0$ and $c \geq 0$, where $f_\theta$ is the p.d.f. of $P_\theta$. If $T$ is a test with $\beta_T(\theta_0) = \beta_{T^*}(\theta_0)$, then for any $\theta > \theta_0$, either $\beta_T(\theta) < \beta_{T^*}(\theta)$ or $T = T^*$ a.s. $P_\theta$.

(v) For any fixed $\theta_1$, $T^*$ is UMP for testing $H_0 : \theta \leq \theta_1$ versus $H_1 : \theta > \theta_1$, with size $\beta_{T^*}(\theta_1)$.

Proof of Theorem 6.2
(i) Consider the hypotheses $\theta = \theta_0$ versus $\theta = \theta_1$ with any $\theta_1 > \theta_0$. A UMP test is given in Theorem 6.1 with $f_j = \text{the p.d.f. of } P_{\theta_j}$, $j = 0, 1$.
Since $P$ has monotone likelihood ratio in $Y(X)$, this UMP test can be chosen to be the same as $T^*$ with possibly different $c$ and $\gamma$ satisfying $\beta_{T^*}(\theta_0) = \alpha$.
Since $T^*$ does not depend on $\theta_1$, it follows from Lemma 6.1 that $T^*$ is UMP for testing the hypothesis $\theta = \theta_0$ versus $H_1$. 

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Proof (continued)

Note that if $T_*$ is UMP for testing $\theta = \theta_0$ versus $H_1$, then it is UMP for testing $H_0$ versus $H_1$, provided that $\beta_{T_*}(\theta) \leq \alpha$ for all $\theta \leq \theta_0$, i.e., the size of $T_*$ is $\alpha$.

But this follows from Lemma 6.3, i.e., $\beta_{T_*}(\theta)$ is nondecreasing in $\theta$.

(ii) See Exercise 2 in §6.6.

(iii) The result can be proved using Theorem 6.1 with all inequalities reversed.

(iv) The proof for (iv) is left as an exercise.

(v) The proof for (v) is similar to that of (i).

Corollary 6.1 (one-parameter exponential families)

Suppose that $X$ has a p.d.f. in a one-parameter exponential family with $\eta$ being a strictly monotone function of $\theta$. If $\eta$ is increasing, then $T_*$ given by Theorem 6.2 is UMP for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\gamma$ and $c$ are determined by $\beta_{T_*}(\theta_0) = \alpha$.

If $\eta$ is decreasing or $H_0 : \theta \geq \theta_0$ ($H_1 : \theta < \theta_0$), the result is still valid by reversing inequalities in the definition of $T_*$. 
Example 6.6
Let $X_1, \ldots, X_n$ be i.i.d. from the $N(\mu, \sigma^2)$ distribution with an unknown $\mu \in \mathbb{R}$ and a known $\sigma^2$.
Consider $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$, where $\mu_0$ is a fixed constant.
The p.d.f. of $X = (X_1, \ldots, X_n)$ is from a one-parameter exponential family with $Y(X) = \bar{X}$ and $\eta(\mu) = n\mu/\sigma^2$.
By Corollary 6.1 and the fact that $\bar{X}$ is $N(\mu, \sigma^2/n)$, the UMP test is $T^*(X) = I(c_\alpha, \infty)(\bar{X})$, where $c_\alpha = \sigma z_{1-\alpha}/\sqrt{n} + \mu_0$ and $z_a = \Phi^{-1}(a)$.

Discussion
To derive a UMP test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ when $X$ has a p.d.f. in a one-parameter exponential family, it is essential to know the distribution of $Y(X)$.
Typically, a nonrandomized test can be obtained if the distribution of $Y$ is continuous; otherwise UMP tests are randomized.
Example 6.8

Let $X_1, \ldots, X_n$ be i.i.d. random variables from the Poisson distribution $P(\theta)$ with an unknown $\theta > 0$.

The p.d.f. of $X = (X_1, \ldots, X_n)$ is from a one-parameter exponential family with $Y(X) = \sum_{i=1}^{n} X_i$ and $\eta(\theta) = \log \theta$.

Note that $Y$ has the Poisson distribution $P(n\theta)$.

By Corollary 6.1, a UMP test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ is given by Theorem 6.2 with $c$ and $\gamma$ satisfying

$$\alpha = \sum_{j=c+1}^{\infty} \frac{e^{n\theta_0}(n\theta_0)^j}{j!} + \gamma \frac{e^{n\theta_0}(n\theta_0)^c}{c!}.$$ 

Example 6.9

Let $X_1, \ldots, X_n$ be i.i.d. random variables from the uniform distribution $U(0, \theta)$, $\theta > 0$.

Consider the hypotheses $H_0 : \theta \leq \theta_0$ and $H_1 : \theta > \theta_0$.

The p.d.f. of $X = (X_1, \ldots, X_n)$ is in a family with monotone likelihood ratio in $Y(X) = X_{(n)}$ (Example 6.4).
Example 6.9 (continued)

By Theorem 6.2, a UMP test is $T_\ast$. Since $X_n$ has the Lebesgue p.d.f. $n\theta^{-n}x^{n-1}I_{(0,\theta)}(x)$, the UMP test $T_\ast$ is nonrandomized and

$$\alpha = \beta_{T_\ast}(\theta_0) = \frac{n}{\theta_0^n} \int_0^{\theta_0} x^{n-1} dx = 1 - \frac{c^n}{\theta_0^n}.$$ 

Hence $c = \theta_0(1 - \alpha)^{1/n}$.

The power function of $T_\ast$ when $\theta > \theta_0$ is

$$\beta_{T_\ast}(\theta) = \frac{n}{\theta^n} \int_0^\theta x^{n-1} dx = 1 - \frac{\theta_0^n(1 - \alpha)}{\theta^n}.$$ 

In this problem, however, UMP tests are not unique. (Note that the condition $P_\theta(f_\theta(X) = cf_{\theta_0}(X)) = 0$ in Theorem 6.2(iv) is not satisfied.)

It can be shown (exercise) that the following test is also UMP with size $\alpha$:

$$T(X) = \begin{cases} 1 & X_n > \theta_0 \\ \alpha & X_n \leq \theta_0 \end{cases}$$
Proposition 6.1 (Generalized Neyman-Pearson lemma)

Let \( f_1, \ldots, f_{m+1} \) be Borel functions on \( \mathcal{R}^p \) integrable w.r.t. a \( \sigma \)-finite \( \nu \).

For given constants \( t_1, \ldots, t_m \), let \( \mathcal{T} \) be the class of Borel functions \( \phi \) (from \( \mathcal{R}^p \) to \([0, 1]\)) satisfying

\[
\int \phi f_i d\nu \leq t_i, \quad i = 1, \ldots, m, \tag{1}
\]

and \( \mathcal{T}_0 \) be the set of \( \phi \)'s in \( \mathcal{T} \) satisfying (1) with all inequalities replaced by equalities. If there are constants \( c_1, \ldots, c_m \) such that

\[
\phi_*(x) = \begin{cases} 
1 & f_{m+1}(x) > c_1 f_1(x) + \cdots + c_m f_m(x) \\
0 & f_{m+1}(x) < c_1 f_1(x) + \cdots + c_m f_m(x)
\end{cases}
\]

is a member of \( \mathcal{T}_0 \), then \( \phi_\ast \) maximizes \( \int \phi f_{m+1} d\nu \) over \( \phi \in \mathcal{T}_0 \).

If \( c_i \geq 0 \) for all \( i \), then \( \phi_\ast \) maximizes \( \int \phi f_{m+1} d\nu \) over \( \phi \in \mathcal{T} \).

The proof is left as an exercise.

The result is useful for finding optimal tests for two sided hypotheses.

The existence of constants $c_i$’s in $\phi_\ast$ is considered in the following lemma whose proof can be found in Lehmann (1986, pp. 97-99).

**Lemma 6.2**

Let $f_1, \ldots, f_m$ and $\nu$ be given by Proposition 6.1. Then the set $M = \{ (\int \phi f_1 d\nu, \ldots, \int \phi f_m d\nu) : \phi \text{ is from } \mathcal{R}^p \text{ to } [0, 1] \}$ is convex and closed. If $(t_1, \ldots, t_m)$ is an interior point of $M$, then there exist constants $c_1, \ldots, c_m$ such that the function $\phi_\ast$ defined in Proposition 6.1 is in $\mathcal{T}_0$.

**Two-sided hypotheses**

The following hypotheses are called two-sided hypotheses:

\begin{align*}
H_0 : \theta &\leq \theta_1 \text{ or } \theta \geq \theta_2 & \text{ versus } H_1 : \theta_1 < \theta < \theta_2, & \quad (2) \\
H_0 : \theta_1 \leq \theta \leq \theta_2 & \text{ versus } H_1 : \theta < \theta_1 \text{ or } \theta > \theta_2, & \quad (3) \\
H_0 : \theta = \theta_0 & \text{ versus } H_1 : \theta \neq \theta_0, & \quad (4)
\end{align*}

where $\theta_0, \theta_1,$ and $\theta_2$ are given constants and $\theta_1 < \theta_2$. 
Theorem 6.3 (UMP tests for two-sided hypotheses)

Suppose that $X$ has a p.d.f. w.r.t. a $\sigma$-finite measure in a one-parameter exponential family:

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x).$$

(i) For testing hypotheses (2), a UMP test of size $\alpha$ is

$$T^*(X) = \begin{cases} 
1 & c_1 < Y(X) < c_2 \\
\gamma_i & Y(X) = c_i, \ i = 1, 2 \\
0 & Y(X) < c_1 \text{ or } Y(X) > c_2,
\end{cases} \quad (5)$$

$c_i$’s and $\gamma_i$’s are determined by

$$\beta_{T^*}(\theta_1) = \beta_{T^*}(\theta_2) = \alpha. \quad (6)$$

(ii) $T^*$ minimizes $\beta_T(\theta)$ over $\theta < \theta_1$, $\theta > \theta_2$ and $T$ satisfying (6).

(iii) If $T^*$ and $T^{**}$ are two tests satisfying (5) and $\beta_{T^*}(\theta_1) = \beta_{T^{**}}(\theta_1)$ and if the region $\{T^{**} = 1\}$ is to the right of $\{T^* = 1\}$, then $\beta_{T^*}(\theta) < \beta_{T^{**}}(\theta)$ for $\theta > \theta_1$ and $\beta_{T^*}(\theta) > \beta_{T^{**}}(\theta)$ for $\theta < \theta_1$. If both $T^*$ and $T^{**}$ satisfy (5) and (6), then $T^* = T^{**}$ a.s. $P$. 
Proof

(i) Since \( Y \) is sufficient for \( \theta \), we only need to consider tests of the form \( T(Y) \).

By Theorem 2.1, the distribution of \( Y \) has a p.d.f.

\[
g_\theta(y) = \exp\{\eta(\theta)y - \xi(\theta)\} \tag{7}\]

Let \( \theta_1 < \theta_3 < \theta_2 \).

Consider the problem of testing \( \theta = \theta_1 \) or \( \theta = \theta_2 \) versus \( \theta = \theta_3 \).

\((\alpha,\alpha)\) is an interior point of the set of all points \((\beta_T(\theta_1),\beta_T(\theta_2))\) as \( T \) ranges over all tests of the form \( T(Y) \).

By (7) and Lemma 6.2, there are constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) such that

\[
T_*(Y) = \begin{cases} 
1 & a_1 e^{b_1 Y} + a_2 e^{b_2 Y} < 1 \\
0 & a_1 e^{b_1 Y} + a_2 e^{b_2 Y} > 1 
\end{cases}
\]

satisfies (6), where \( a_i = \tilde{c}_i e^{\xi(\theta_3) - \xi(\theta_i)} \) and \( b_i = \eta(\theta_i) - \eta(\theta_3) \), \( i = 1, 2 \).

Clearly \( a_i \)'s cannot both be \( \leq 0 \).
If one of the \( a_i \)'s is \( \leq 0 \) and the other is \( > 0 \), then \( a_1 e^{b_1 Y} + a_2 e^{b_2 Y} \) is strictly monotone (since \( b_1 < 0 < b_2 \)) and

\[
T_*( \text{ or } 1 - T_*) = \begin{cases} 
1 & Y(X) > c \\
\gamma & Y(X) = c \\
0 & Y(X) < c, 
\end{cases}
\]

which has a strictly monotone power function and, therefore, cannot satisfy \( \beta_T(\theta_1) = \beta_T(\theta_2) = \alpha \). Thus, both \( a_i \)'s are positive. The function \( a_1 e^{b_1 Y} + a_2 e^{b_2 Y} \) is convex.

\( a_1 e^{b_1 Y} + a_2 e^{b_2 Y} < 1 \) is equivalent to \( c_1 < Y < c_2 \) for some \( c_1 \) and \( c_2 \).

Then, \( T_* \) is of the form (5) and it follows from Proposition 6.1 that \( T_* \) is UMP for testing \( \theta = \theta_1 \) or \( \theta = \theta_2 \) versus \( \theta = \theta_3 \).

Since \( T_* \) does not depend on \( \theta_3 \), it follows from Lemma 6.1 that \( T_* \) is UMP for testing \( \theta = \theta_1 \) or \( \theta = \theta_2 \) versus \( H_1 \).

To show that \( T_* \) is a UMP test of size \( \alpha \) for testing \( H_0 \) versus \( H_1 \), it remains to show that \( \beta_{T_*}(\theta) \leq \alpha \) for \( \theta \leq \theta_1 \) or \( \theta \geq \theta_2 \), which follows from part (ii) of the theorem by comparing \( T_* \) with the test \( T(Y) \equiv \alpha \).
Example 6.10

Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, 1)$. By Theorem 6.3, a UMP test for testing (2) is $T^*(X) = I_{(c_1, c_2)}(\bar{X})$, where $c_i$’s are determined by

$$\Phi\left(\sqrt{n}(c_2 - \theta_1)\right) - \Phi\left(\sqrt{n}(c_1 - \theta_1)\right) = \alpha$$

and

$$\Phi\left(\sqrt{n}(c_2 - \theta_2)\right) - \Phi\left(\sqrt{n}(c_1 - \theta_2)\right) = \alpha.$$

When the distribution of $X$ is not from a one-parameter exponential family, UMP tests for hypotheses (2) exist in some cases (see Exercises 17 and 26). Unfortunately, a UMP test does not exist in general for testing hypotheses (3) or (4) (Exercises 28 and 29). A key reason for this phenomenon is that UMP tests for testing one-sided hypotheses do not have level $\alpha$ for testing (2); but they are of level $\alpha$ for testing (3) or (4) and there does not exist a single test more powerful than all tests that are UMP for testing one-sided hypotheses.