1. Let $h(x_1, x_2, x_3) = I_{(-\infty, 0)}(x_1 + x_2 + x_3)$. Find $h_k$ and $\zeta_k$, $k = 1, 2, 3$, for the U-statistic with kernel $h$ based on independent random variables $X_1, X_2, \ldots, X_n$ with a common cumulative distribution function $F$.

2. Let $X_1, \ldots, X_n$ be a random sample of random variables having finite $E X^2_1$ and $E X^{-2}_1$. Let $\mu = E X_1$ and $\bar{\mu} = E X^{-1}_1$. Find a U-statistic that is an unbiased estimator of $\mu \bar{\mu}$ and derive its variance and asymptotic distribution.

3. Consider the polynomial model

$$X_i = \beta_0 + \beta_1 t_i + \beta_2 t^2_i + \beta_3 t^3_i + \epsilon_i, i = 1, \ldots, n,$$

where $\epsilon_i$’s are independent and identically distributed random variables with mean 0. Suppose that $n = 12$, $t_i = -1$, $i = 1, 4, 5, \ldots, 8$, and $t_i = 1$, $i = 9, \ldots, 12$. Show whether the following parameters are estimable (i.e., they can be unbiasedly estimated): $\beta_0 + \beta_2$, $\beta_1$, $\beta_0 - \beta_1$, $\beta_1 + \beta_3$, and $\beta_0 + \beta_1 + \beta_2 + \beta_3$.

4. Assume that $X$ is a random $n$-vector from the multivariate normal distribution $N(Z\beta, \sigma^2 I_n)$, where $Z$ is an $n \times p$ known matrix of rank $r \leq p < n$, $\beta$ is a $p$-vector of unknown parameters, $I_n$ is the identity matrix of order $n$, and $\sigma^2 > 0$ is unknown. Find the UMVUEs of $(l^T \beta)^2$, $l^T \beta/\sigma$ and $(l^T \beta/\sigma)^2$ for an estimable $l^T \beta$.

5. Consider the linear model $X = Z\beta + \epsilon$, where $Z$ is a known $n \times p$ matrix, $\beta$ is a $p$-vector of unknown parameters, and $\epsilon$ is a random $n$-vector whose components are independent and identically distributed with mean 0 and variance $\sigma^2$. Let $X_i$ be the $i$th component of $X$, $Z_i$ be the $i$th row of $Z$, $h_{ij}$ be the $(i, j)$th element of $Z(Z^T Z)^{-1} Z^T$, $h_i = h_{ii}$, $\hat{\beta}$ be an LSE of $\beta$, and $\hat{X}_i = Z_i^T \hat{\beta}$. Show that

(i) $\text{Var}(\hat{X}_i) = \sigma^2 h_i$;

(ii) $\text{Var}(X_i - \hat{X}_i) = \sigma^2 (1 - h_i)$;

(iii) $\text{Cov}(\hat{X}_i, \hat{X}_j) = \sigma^2 h_{ij}$;

(iv) $\text{Cov}(X_i - \hat{X}_i, X_j - \hat{X}_j) = -\sigma^2 h_{ij}, i \neq j$;
(v) \( \text{Cov}(\hat{X}_i, X_j - \hat{X}_j) = 0. \)

6. Consider the linear model \( X = Z\beta + \epsilon, \) where \( Z \) is a known \( n \times p \) matrix, \( \beta \) is a \( p \)-vector of unknown parameters, and \( \epsilon \) is a random \( n \)-vector whose components are independent and identically distributed with mean 0 and variance \( \sigma^2. \) Let \( Z = (Z_1, Z_2) \) and \( \beta = (\beta_1^\tau, \beta_2^\tau)^\tau, \) where \( Z_j \) is \( n \times p_j \) and \( \beta_j \) is a \( p_j \)-vector, \( j = 1, 2. \) Assume that \( (Z_1^\tau Z_1)^{-1} \) and \( [Z_2^\tau Z_2 - Z_2^\tau Z_1 (Z_1^\tau Z_1)^{-1} Z_1^\tau Z_2]^{-1} \) exist.

(i) Derive the LSE of \( \beta \) in terms of \( Z_1, Z_2, \) and \( X. \)

(ii) Let \( \hat{\beta} = (\hat{\beta}_1^\tau, \hat{\beta}_2^\tau)^\tau \) be the LSE in (i). Calculate the covariance between \( \hat{\beta}_1 \) and \( \hat{\beta}_2. \)

(iii) Suppose that it is known that \( \beta_2 = 0. \) Let \( \tilde{\beta}_1 \) be the LSE of \( \beta_1 \) under the reduced model \( X = Z_1 \beta_1 + \epsilon. \) Show that, for any \( l \in \mathbb{R}^{p_1}, l^\tau \beta \) is better than \( l^\tau \hat{\beta} \) in terms of their variances.