1. Show that \( \{P_\theta : \theta \in \Theta\} \) is an exponential family and find its canonical form and natural parameter space, when

(i) \( P_\theta \) is the Poisson distribution \( P(\theta) : \theta \in \Theta = (0, \infty) \);

(ii) \( P_\theta \) is the negative binomial distribution \( NB(\theta, r) \) with a fixed \( r \), \( \theta \in \Theta = (0, 1) \);

(iii) \( P_\theta \) is the exponential distribution \( E(a, \theta) \) with a fixed \( a \), \( \theta \in \Theta = (0, \infty) \);

(iv) \( P_\theta \) is the gamma distribution \( \Gamma(\alpha, \gamma) \), \( \theta = (\alpha, \gamma) \in \Theta = (0, \infty) \otimes (0, \infty) \);

(v) \( P_\theta \) is the beta distribution \( B(\alpha, \beta) \), \( \theta = (\alpha, \beta) \in \Theta = (0, 1) \otimes (0, 1) \);

(vi) \( P_\theta \) is the Weibull distribution \( W(\alpha, \theta) \) with a fixed \( \alpha > 0 \), \( \theta \in \Theta = (0, \infty) \).

Solution:

(i) The p.d.f. of the Poisson distribution can be expressed as

\[
\theta^x e^{-\theta}/x! = \exp\{x \ln \theta - \theta\}I_{(0,1,2,...)}(x),
\]

which has the form of an exponential family. \( \eta = \ln \theta \) is the natural parameter and the natural parameter space is \((-\infty, \infty)\).

(iii) The exponential distributions with fixed \( a \) has the p.d.f.

\[
\theta^{-1} e^{-(x-a)/\theta}I_{(a,\infty)}(x) = \exp\{-\theta^{-1}(x-a)-\ln \theta\}I_{(a,\infty)}(x) = \exp\{\eta(x-a)+\ln(-\eta)\}I_{(a,\infty)}(x)
\]

which is in the form of an exponential family. The natural parameter is \( \eta = -\theta^{-1} \). The natural parameter space is \( \Xi = (-\infty, 0) \).

(v) The beta distribution has the p.d.f.

\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}I_{(0,1)}(x) = \exp\{\alpha \ln x + \beta \ln(1-x) + \ln \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\}[x(1-x)]^{-1}I_{(0,1)}(x),
\]

which is in the canonical form of an exponential family. The natural parameter is \( \theta = (\alpha, \beta) \). The natural parameter space is \( \Theta = (0, 1) \otimes (0, 1) \).
2. Show that \( \{ P_\theta : \theta \in \Theta \} \) is not an exponential family, when

(i) \( P_\theta \) is the exponential distribution \( E(a, \theta) \) with two unknown parameters \( a \), and \( \theta \);

(ii) \( P_\theta \) is the negative binomial distribution \( NB(\theta, r) \) with two unknown parameters \( r \) and \( \theta \).

Solution:

(i) If \( E(a, \theta) \) is an exponential family, then \( E(a, \theta) \) has a positive density

\[
\exp\{ \eta(a, \theta)^T T(x) - \xi(a, \theta) \}
\]

with respect to a non-zero measure \( \nu \).

Consider the interval \((-\infty, t)\) for any \( t \in \mathbb{R} \). There is an \( a \in \mathbb{R} \) such that \( a > t \), hence \( P_{(a, \theta)}[(-\infty, t)] = 0 \). This together with (1) implies that \( \nu[(-\infty, t)] = 0 \). Since \( t \) is arbitrary, \( \nu \) must be a zero measure, which is a contradiction.

(ii) The proof is the same.

The method above is a general method for proving that any family of distributions with domain (the range of nonzero density function) depending on unknown parameters is not an exponential family.
3. Show that the family of double exponential distributions $DE(\mu, \theta)$ with two unknown parameters $\mu$ and $\theta$ is not an exponential family, but the family with a fixed $\mu$ and an unknown parameter $\theta$ is an exponential family.

Solution:

(i) Without loss of generality assume $\theta = 1$ is known. By a counter-proof, assume that $DE(\mu, \theta)$ is an exponential family, we are going to arrive at a contradiction. If $DE(\mu, \theta)$ is an exponential family, then there exist $p$-dimensional Borel functions $T(X)$ and $\eta(\mu)$ ($p \geq 1$) and one-dimensional Borel functions $h(X)$ and $\xi(\mu)$ such that

$$
\frac{1}{2} e^{-|x-\mu|} = e^{\eta(\mu)^{\top} T(x) - \xi(\mu) h(x)}
$$

for any $x$ and $\mu$. Let $X = (X_1, \ldots, X_n)$ be a random sample from $DE(\mu, \theta)$, where $n > p$. Let $T_n(X) = \sum_{i=1}^{n} T(X_i)$ and $h_n(X) = \prod_{i=1}^{n} h(X_i)$. Then the joint Lebesgue density of $X$ is

$$
\frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i-\mu|} = e^{\eta(\mu)^{\top} T_n(x) - n\xi(\mu) h_n(x)}
$$

for any $x = (x_1, \ldots, x_n)$ and $\mu$, which implies that

$$
\sum_{i=1}^{n} |x_i| - \sum_{i=1}^{n} |x_i - \mu| = \tilde{\eta}(\mu)^{\top} T_n(x) - n\tilde{\xi}(\mu)
$$

for any $x$ and $\mu$, where $\tilde{\eta}(\mu) = \eta(\mu) - \eta(0)$ and $\tilde{\xi}(\mu) = \xi(\mu) - \xi(0)$. Define $\psi_\mu(x) = \sum_{i=1}^{n} |x_i| - \sum_{i=1}^{n} |x_i - \mu|$.

We conclude that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ such that $T_n(x) = T_n(y)$, then $\psi_\mu(x) = \psi_\mu(y)$ for all $\mu$, which implies that vector of the ordered $x_i$’s is the same as the vector of the ordered $y_i$’s.

On the other hand, we may choose real numbers $\mu_1, \ldots, \mu_p$ such that $\tilde{\eta}(\mu_i), i = 1, ..., p$, are linearly independent vectors. Since

$$
\psi_{\mu_i}(x) = \tilde{\eta}(\mu_i)^{\top} T_n(x) - n\tilde{\xi}(\mu_i)
$$

for any $x$, $T_n(x)$ is then a function of the $p$ functions $\psi_{\mu_i}(x), i = 1, ..., p$. Since $n > p$, it can be shown that there exist $x$ and $y$ in $\mathbb{R}^n$ such that $\psi_{\mu_i}(x) = \psi_{\mu_i}(y), i = 1, ..., p$, (which implies $T_n(x) = T_n(y)$), but the vector of ordered $x_i$’s is not the same as the vector of ordered $y_i$’s. This contradicts the previous conclusion. Hence, $P$ is not an exponential family.
The same method can be used to show that the family of Cauchy distributions with unknown $\mu$ parameter, of Weibull distributions with unknown $\alpha$ parameter is not an exponential family.

(ii) The proof is trivial.
4. Consider the multinomial distribution with p.d.f. given by
\[ f(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1! \cdots x_k!} \theta_1^{x_1} \cdots \theta_{k-1}^{x_{k-1}} \theta_k^{x_k}, \]
where \( x_j \)'s are integers satisfying \( \sum_{j=1}^k x_j = n \) and \( \theta_j > 0, \sum_{j=1}^k \theta_j = 1 \).

(i) Show that the multinomial distribution is an exponential family with \( \theta = (\theta_1, \ldots, \theta_k) \), but it does not have a full rank.

(ii) Provide a re-parameterization of the family such that, with the re-parameterized parameter space, the multinomial distribution is an full rank exponential family.

Solution:

(i) The p.d.f. can be expressed as
\[ f(x_1, x_2, \ldots, x_k) = \exp\left\{ \sum_{j=1}^k \ln(\theta_j) x_j \right\} \frac{n!}{x_1! \cdots x_k!}, \]
which is of the form of an exponential family with parameter space \( \{ (\theta_1, \ldots, \theta_k) : \theta_j > 0, j = 1, \ldots, k, \sum_{j=1}^k \theta_j = 1 \} \). However, the parameter space does not contain a open set in the space \( \mathbb{R}^k \). Hence it does not have full rank.

(ii) Re-parameterize the family by \( \vartheta = (\theta_1, \ldots, \theta_{k-1}) \) where \( \theta_j > 0, j = 1, \ldots, k-1, \sum_{j=1}^{k-1} \theta_j < 1 \). With the new parameterization, the p.d.f. becomes
\[
\begin{align*}
  f(x_1, x_2, \ldots, x_k) &= \exp\left\{ \sum_{j=1}^{k-1} \ln(\theta_j) x_j + \ln(1 - \sum_{j=1}^{k-1} \theta_j)(n - \sum_{j=1}^{k-1} x_j) \right\} \frac{n!}{x_1! \cdots x_k!}, \\
  &= \exp\left\{ \sum_{j=1}^{k-1} x_j \ln \frac{\theta_j}{1 - \sum_{j=1}^{k-1} \theta_j} + n \ln(1 - \sum_{j=1}^{k-1} \theta_j) \right\} \frac{n!}{x_1! \cdots x_k!}.
\end{align*}
\]

The new parameter space itself is an open set in \( \mathbb{R}^{k-1} \). Hence the family is of full rank \( k-1 \).
5. Let $X_1, \ldots, X_n$ be i.i.d. samples from a population $P \in \mathcal{P}$ where $\mathcal{P}$ is any family of distributions. Show that $T(X) = (X_1^\downarrow, \ldots, X_n^\downarrow)$ is sufficient for $\mathcal{P}$, where $X_k^\downarrow$ is the $k$-th smallest order statistic.

Solution:

Given $(X_1^\downarrow, \ldots, X_n^\downarrow)$, $(X_1, \ldots, X_n)$ has an equal chance to assume any permutations of $(X_1^\downarrow, \ldots, X_n^\downarrow)$. Hence, the conditional probability of $(X_1, \ldots, X_n)$ is

$$P(X_1 = x_1, \ldots, X_n = x_n) = \frac{1}{n!},$$

where $(x_1, \ldots, x_n)$ is any permutation of $(X_1^\downarrow, \ldots, X_n^\downarrow)$. No matter what family is assumed, the conditional distribution does not depend on any unknowns of the family. Hence $T(X)$ is sufficient by the definition.