ST5215: Advanced Statistical Theory

Chen Zehua

Department of Statistics & Applied Probability

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Examples of minimal sufficiency

Example 2.14

Let $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$ be an exponential family with p.d.f.'s

$$f_\theta(x) = \exp\{[\eta(\theta)]^T T(x) - \xi(\theta)\} h(x).$$

By Factorization Theorem, $T(X)$ is sufficient for $\theta \in \Theta$. Suppose that there exists $\Theta_0 = \{\theta_0, \theta_1, \ldots, \theta_p\} \subset \Theta$ such that the vectors

$$\eta_i = \eta(\theta_i) - \eta(\theta_0), \quad i = 1, \ldots, p,$$

are linearly independent in $\mathbb{R}^p$. (This is true if the exponential family is of full rank). Then $T$ is also minimal sufficient.

Solution A: Let $\mathcal{P}_0 = \{f_\theta : \theta \in \Theta_0\}$. Note that the set $\{x : f_\theta(x) > 0\}$ does not depend on $\theta$. It follows from Theorem 2.3(ii) with $f_\infty = f_{\theta_0}$ that

$$S(X) = \left(\exp\{\eta_1^T T(x) - \xi_1\}, \ldots, \exp\{\eta_p^T T(x) - \xi_p\}\right)$$

is minimal sufficient for $\theta \in \Theta_0$. 
Example 2.14 (cont.)

Since $\eta_i$'s are linearly independent, there is a one-to-one measurable function $\psi$ such that $T(X) = \psi(S(X))$ a.s. $\mathcal{P}_0$. Hence, $T$ is minimal sufficient for $\theta \in \Theta_0$. It is easy to see that a.s. $\mathcal{P}_0$ implies a.s. $\mathcal{P}$. Thus, by Theorem 2.3(i), $T$ is minimal sufficient for $\theta \in \Theta$.

**Solution B:** Let $\phi(x, y) = h(x)/h(y)$. Then

$$f_\theta(x) = f_\theta(y)\phi(x, y)$$

$$\Rightarrow \exp\{[\eta(\theta)]^T[T(x) - T(y)]\} = 1$$

$$\Rightarrow T(x) = T(y).$$

Since $T$ is sufficient, by Thorem 2.3 (iii), $T$ is also minimal sufficient.
Example 2.13 (revisited)

Let $X_1, \ldots, X_n$ be i.i.d. random variables form $P_\theta$, the uniform distribution $U(\theta, \theta + 1)$, $\theta \in \mathbb{R}$, $n > 1$.

The joint Lebesgue p.d.f. of $(X_1, \ldots, X_n)$ is

$$f_\theta(x) = \prod_{i=1}^{n} I_{(\theta, \theta+1)}(x_i) = I_{(x_n(1)-1, x_1(1))}(\theta), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $x(i)$ denotes the $i$th smallest value of $x_1, \ldots, x_n$.

Here is another way to show that $T = (X_{(1)}, X_{(n)})$ is minimal sufficient.

Let $\phi(x, y) = 1$. Then

$$f_\theta(x) = f_\theta(y), \text{ for all } \theta$$

$$\Rightarrow I_{(x_n(1)-1, x_1(1))}(\theta) = I_{(y_n(1)-1, y_1(1))}(\theta) \text{ for all } \theta$$

$$\Rightarrow (x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)}).$$

By Theorem 2.3 (iii), $T = (X_{(1)}, X_{(n)})$ is minimal sufficient.
Remarks

- The sufficiency (and minimal sufficiency) depends on the postulated family $\mathcal{P}$ of populations (statistical models).
- It may not be a useful concept if the proposed statistical model is wrong or at least one has some doubts about the correctness of the proposed model.
- From the examples in this section and some exercises in §2.6, one can find that for a wide variety of models, statistics such as the sample mean $\bar{X}$, the sample variance $S^2$, $(X_{(1)}, X_{(n)})$ in Example 2.11, and the order statistics in Example 2.9 are sufficient.
- Thus, using these statistics for data reduction and summarization does not lose any information when the true model is one of those models but we do not know exactly which model is correct.
- A minimal sufficient statistic is not always the “simplest sufficient statistic”.
- For example, if $\bar{X}$ is minimal sufficient, then so is $(\bar{X}, \exp\{\bar{X}\})$.
Completeness

A statistic $V(X)$ is *ancillary* if its distribution does not depend on the population $P$.

$V(X)$ is *first-order ancillary* if $E[V(X)]$ is independent of $P$.

A trivial ancillary statistic is the constant statistic $V(X) \equiv c \in \mathcal{R}$.

If $V(X)$ is a nontrivial ancillary statistic, then $\sigma(V(X)) \subset \sigma(X)$ is a nontrivial $\sigma$-field that does not contain any information about $P$.

Hence, if $S(X)$ is a statistic and $V(S(X))$ is a nontrivial ancillary statistic, it indicates that $\sigma(S(X))$ contains a nontrivial $\sigma$-field that does not contain any information about $P$ and, hence, the “data” $S(X)$ may be further reduced.

A sufficient statistic $T$ appears to be most successful in reducing the data if no nonconstant function of $T$ is ancillary or even first-order ancillary.

This leads to the notion of completeness.
Definition 2.6 (Completeness)

A statistic $T(X)$ is said to be complete for $P \in \mathcal{P}$ iff, for any Borel function $f$, $E[f(T)] = 0$ for all $P \in \mathcal{P}$ implies $f = 0$ a.s. $P$.

$T$ is said to be boundedly complete iff the previous statement holds for any bounded Borel function $f$.

Remarks

- A complete statistic is boundedly complete.
- If $T$ is complete (or boundedly complete) and $S = \psi(T)$ for a measurable function $\psi$, then $S$ is complete (or boundedly complete).
- Intuitively, a complete and sufficient statistic should be minimal sufficient (Exercise 48).
- A minimal sufficient statistic is not necessarily complete; for example, the minimal sufficient statistic $(X_{(1)}, X_{(n)})$ in Example 2.13 is not complete (Exercise 47).
Proposition 2.1
If $P$ is in an exponential family of full rank with p.d.f.’s given by

$$f_\eta(x) = \exp\left\{\eta^T T(x) - \zeta(\eta)\right\} h(x),$$

then $T(X)$ is complete and sufficient for $\eta \in \Xi$.

Proof
We have shown that $T$ is sufficient.

We now show that $T$ is complete.

Suppose that there is a function $f$ such that $E[f(T)] = 0$ for all $\eta \in \Xi$.

By Theorem 2.1(i),

$$\int f(t) \exp\{\eta^T t - \zeta(\eta)\} d\lambda = 0 \quad \text{for all } \eta \in \Xi,$$

where $\lambda$ is a measure on $(\mathcal{R}^p, \mathcal{B}^p)$. 
Proof (continued)

Let $\eta_0$ be an interior point of $\Xi$. Then

$$
\int f_+(t)e^{\eta^\top t}d\lambda = \int f_-(t)e^{\eta^\top t}d\lambda \quad \text{for all } \eta \in N(\eta_0), \tag{1}
$$

where $N(\eta_0) = \{ \eta \in \mathcal{R}^p : \| \eta - \eta_0 \| < \epsilon \}$ for some $\epsilon > 0$.

In particular,

$$
\int f_+(t)e^{\eta_0^\top t}d\lambda = \int f_-(t)e^{\eta_0^\top t}d\lambda = c.
$$

If $c = 0$, then $f = 0$ a.e. $\lambda$.

If $c > 0$, then $c^{-1}f_+(t)e^{\eta_0^\top t}$ and $c^{-1}f_-(t)e^{\eta_0^\top t}$ are p.d.f.'s w.r.t. $\lambda$ and result (1) implies that their m.g.f.'s are the same in a neighborhood of 0.

By Theorem 1.6(ii), $c^{-1}f_+(t)e^{\eta_0^\top t} = c^{-1}f_-(t)e^{\eta_0^\top t}$, i.e.,

$f = f_+ - f_- = 0$ a.e. $\lambda$.

Hence $T$ is complete.
Example 2.15

Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables having the $N(\mu, \sigma^2)$ distribution, $\mu \in \mathbb{R}$, $\sigma > 0$.

From Example 2.6, the joint p.d.f. of $X_1, \ldots, X_n$ is

$$(2\pi)^{-n/2} \exp \left\{ \eta_1 T_1 + \eta_2 T_2 - n \zeta(\eta) \right\},$$

where $T_1 = \sum_{i=1}^n X_i$, $T_2 = - \sum_{i=1}^n X_i^2$, and $\eta = (\eta_1, \eta_2) = \left( \frac{\mu}{\sigma^2}, \frac{1}{2\sigma^2} \right)$.

Hence, the family of distributions for $X = (X_1, \ldots, X_n)$ is a natural exponential family of full rank ($\Xi = \mathbb{R} \times (0, \infty)$). By Proposition 2.1, $T(X) = (T_1, T_2)$ is complete and sufficient for $\eta$.

Since there is a one-to-one correspondence between $\eta$ and $\theta = (\mu, \sigma^2)$, $T$ is also complete and sufficient for $\theta$. It can be shown that any one-to-one measurable function of a complete and sufficient statistic is also complete and sufficient (exercise).

Thus, $(\bar{X}, S^2)$ is complete and sufficient for $\theta$, where $\bar{X}$ and $S^2$ are the sample mean and sample variance, respectively.
Example 2.16

Let $X_1, \ldots, X_n$ be i.i.d. random variables from $P_{\theta}$, the uniform distribution $U(0, \theta)$, $\theta > 0$. The largest order statistic, $X_{(n)}$, is complete and sufficient for $\theta \in (0, \infty)$.

The sufficiency of $X_{(n)}$ follows from the fact that the joint Lebesgue p.d.f. of $X_1, \ldots, X_n$ is $\theta^{-n} I_{(0,\theta)}(x(n))$.

From Example 2.9, $X_{(n)}$ has the Lebesgue p.d.f. $(nx^{n-1}/\theta^n) I_{(0,\theta)}(x)$. Let $f$ be a Borel function on $[0, \infty)$ such that $E[f(X_{(n)})] = 0$ for all $\theta > 0$. Then

$$\int_0^{\theta} f(x)x^{n-1}dx = 0 \quad \text{for all } \theta > 0.$$

Let $G(\theta)$ be the left-hand side of the previous equation.

Applying the result of differentiation of an integral (see, e.g., Royden (1968, §5.3)), we obtain that $G'(\theta) = f(\theta)\theta^{n-1}$ a.e. $m_+$, where $m_+$ is the Lebesgue measure on $([0, \infty), \mathcal{B}_{[0,\infty)})$. Since $G(\theta) = 0$ for all $\theta > 0$, $f(\theta)\theta^{n-1} = 0$ a.e. $m_+$ and, hence, $f(x) = 0$ a.e. $m_+$. Therefore, $X_{(n)}$ is complete and sufficient for $\theta \in (0, \infty)$. 
Example 2.17
In Example 2.12, we showed that the order statistics
\( T(X) = (X_{(1)}, ..., X_{(n)}) \) of i.i.d. random variables \( X_1, ..., X_n \) is sufficient for \( P \in \mathcal{P} \), where \( \mathcal{P} \) is the family of distributions on \( \mathcal{R} \) having Lebesgue p.d.f.’s.
We now show that \( T(X) \) is also complete for \( P \in \mathcal{P} \).
Let \( \mathcal{P}_0 \) be the family of Lebesgue p.d.f.’s of the form
\[
f(x) = C(\theta_1, ..., \theta_n) \exp\{-x^{2n} + \theta_1 x + \theta_2 x^2 + \cdots + \theta_n x^n\},
\]
where \( \theta_j \in \mathcal{R} \) and \( C(\theta_1, ..., \theta_n) \) is a normalizing constant such that
\[
\int f(x) \, dx = 1.
\]
Then \( \mathcal{P}_0 \subset \mathcal{P} \) and \( \mathcal{P}_0 \) is an exponential family of full rank.
Note that the joint distribution of \( X = (X_1, ..., X_n) \) is also in an exponential family of full rank.
Thus, by Proposition 2.1, \( U = (U_1, ..., U_n) \) is a complete statistic for \( P \in \mathcal{P}_0 \), where \( U_j = \sum_{i=1}^{n} X_i^j \).
Since a.s. \( \mathcal{P}_0 \) implies a.s. \( \mathcal{P} \), \( U(X) \) is also complete for \( P \in \mathcal{P} \).
Example 2.17 (continued)

The result follows if we can show that there is a one-to-one correspondence between $T(X)$ and $U(X)$.

Let $V_1 = \sum_{i=1}^{n} X_i$, $V_2 = \sum_{i<j} X_i X_j$, $V_3 = \sum_{i<j<k} X_i X_j X_k$, ..., $V_n = X_1 \cdots X_n$.

From the identities

$U_k - V_1 U_{k-1} + V_2 U_{k-2} - \cdots + (-1)^{k-1} V_{k-1} U_1 + (-1)^k k V_k = 0,$

$k = 1, \ldots, n$, there is a one-to-one correspondence between $U(X)$ and $V(X) = (V_1, \ldots, V_n)$.

From the identity

$(t - X_1) \cdots (t - X_n) = t^n - V_1 t^{n-1} + V_2 t^{n-2} - \cdots + (-1)^n V_n,$

there is a one-to-one correspondence between $V(X)$ and $T(X)$.

This completes the proof and, hence, $T(X)$ is sufficient and complete for $P \in \mathcal{P}$.

In fact, both $U(X)$ and $V(X)$ are sufficient and complete for $P \in \mathcal{P}$.
The relationship between an ancillary statistic and a complete and sufficient statistic is characterized in the following result.

**Theorem 2.4 (Basu’s theorem)**

Let $V$ and $T$ be two statistics of $X$ from a population $P \in \mathcal{P}$. If $V$ is ancillary and $T$ is boundedly complete and sufficient for $P \in \mathcal{P}$, then $V$ and $T$ are independent w.r.t. any $P \in \mathcal{P}$. 

Proof:

Let $B$ be an event on the range of $V$. Since $V$ is ancillary, $P(V^{-1}(B))$ is a constant. As $T$ is sufficient, $E[I_B(V) | T]$ is a function of $T$ (not dependent on $P$).

Because $E\{E[I_B(V) | T] - P(V^{-1}(B))\} = 0$ for all $P \in \mathcal{P}$, by the bounded completeness of $T$, $P(V^{-1}(B) | T) = E[I_B(V) | T] = P(V^{-1}(B))$ a.s. $P$. 


The relationship between an ancillary statistic and a complete and sufficient statistic is characterized in the following result.

**Theorem 2.4 (Basu’s theorem)**

Let $V$ and $T$ be two statistics of $X$ from a population $P \in \mathcal{P}$. If $V$ is ancillary and $T$ is boundedly complete and sufficient for $P \in \mathcal{P}$, then $V$ and $T$ are independent w.r.t. any $P \in \mathcal{P}$.

**Proof**

Let $B$ be an event on the range of $V$.

Since $V$ is ancillary, $P(V^{-1}(B))$ is a constant.

As $T$ is sufficient, $E[I_B(V)|T]$ is a function of $T$ (not dependent on $P$).

Because

$$E\{E[I_B(V)|T] - P(V^{-1}(B))\} = 0 \quad \text{for all } P \in \mathcal{P},$$

by the bounded completeness of $T$,

$$P(V^{-1}(B)|T) = E[I_B(V)|T] = P(V^{-1}(B)) \quad \text{a.s. } \mathcal{P}$$
Proof (continued)
Let $A$ be an event on the range of $T$.
Then
\[ P(T^{-1}(A) \cap V^{-1}(B)) = E\{E[I_A(T)I_B(V)|T]\} = E\{I_A(T)E[I_B(V)|T]\} \]
\[ = E\{I_A(T)P(V^{-1}(B))\} = P(T^{-1}(A))P(V^{-1}(B)). \]
Hence $T$ and $V$ are independent w.r.t. any $P \in \mathcal{P}$.

Remark
Basu’s theorem is useful in proving the independence of two statistics.
Proof (continued)

Let $A$ be an event on the range of $T$. Then

$$P(T^{-1}(A) \cap V^{-1}(B)) = E\{E[I_A(T)I_B(V) | T]\} = E\{I_A(T) E[I_B(V) | T]\} = E\{I_A(T) P(V^{-1}(B))\} = P(T^{-1}(A)) P(V^{-1}(B)).$$

Hence $T$ and $V$ are independent w.r.t. any $P \in \mathcal{P}$.

Remark

Basu’s theorem is useful in proving the independence of two statistics.

Example 2.18

Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables having the $N(\mu, \sigma^2)$ distribution, with $\mu \in \mathcal{R}$ and a known $\sigma > 0$.

It can be easily shown that the family $\{N(\mu, \sigma^2) : \mu \in \mathcal{R}\}$ is an exponential family of full rank with natural parameter $\eta = \mu/\sigma^2$.

By Proposition 2.1, the sample mean $\bar{X}$ is complete and sufficient for $\eta$ (and $\mu$).
Example 2.18 (continued)

Let $S^2$ be the sample variance.

Since $S^2 = (n - 1)^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2$, where $Z_i = X_i - \mu$ is $N(0, \sigma^2)$ and $\bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i$, $S^2$ is an ancillary statistic ($\sigma^2$ is known).

By Basu’s theorem, $\bar{X}$ and $S^2$ are independent w.r.t. $N(\mu, \sigma^2)$ with $\mu \in \mathcal{R}$.

Since $\sigma^2$ is arbitrary, $\bar{X}$ and $S^2$ are independent w.r.t. $N(\mu, \sigma^2)$ for any $\mu \in \mathcal{R}$ and $\sigma^2 > 0$. 

Using the independence of $\bar{X}$ and $S^2$, we now show that $(n - 1) S^2 / \sigma^2$ has the chi-square distribution $\chi^2_{n-1}$.

Note that $n (\bar{X} - \mu \sigma)^2 + (n - 1) S^2 \sigma^2 = n \sum_{i=1}^{n} (X_i - \mu \sigma)^2$. 


Example 2.18 (continued)

Let $S^2$ be the sample variance.

Since $S^2 = (n - 1)^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2$, where $Z_i = X_i - \mu$ is $N(0, \sigma^2)$ and $\bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i$, $S^2$ is an ancillary statistic ($\sigma^2$ is known).

By Basu’s theorem, $\bar{X}$ and $S^2$ are independent w.r.t. $N(\mu, \sigma^2)$ with $\mu \in \mathcal{R}$.

Since $\sigma^2$ is arbitrary, $\bar{X}$ and $S^2$ are independent w.r.t. $N(\mu, \sigma^2)$ for any $\mu \in \mathcal{R}$ and $\sigma^2 > 0$.

Using the independence of $\bar{X}$ and $S^2$, we now show that $(n - 1)S^2/\sigma^2$ has the chi-square distribution $\chi^2_{n-1}$.

Note that

$$
n \left( \frac{\bar{X} - \mu}{\sigma} \right)^2 + \frac{(n - 1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2.
$$
Example 2.18 (continued)

From the properties of the normal distributions, \( n(\bar{X} - \mu)^2/\sigma^2 \) has the chi-square distribution \( \chi^2_1 \) with the m.g.f. \((1 - 2t)^{-1/2}\) and \( \sum_{i=1}^n (X_i - \mu)^2/\sigma^2 \) has the chi-square distribution \( \chi^2_n \) with the m.g.f. \( (1 - 2t)^{-n/2} \), \( t < 1/2 \).

By the independence of \( \bar{X} \) and \( S^2 \), the m.g.f. of \( (n - 1)S^2/\sigma^2 \) is

\[
(1 - 2t)^{-n/2}/(1 - 2t)^{-1/2} = (1 - 2t)^{-(n-1)/2}
\]

for \( t < 1/2 \).

This is the m.g.f. of the chi-square distribution \( \chi^2_{n-1} \) and, therefore, the result follows.