ST5215: Advanced Statistical Theory (I)

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Theorem 1.8

(i) If $X_n \to_{a.s.} X$, then $X_n \to_p X$. (The converse is not true.)

(ii) If $X_n \to_{L_r} X$ for an $r > 0$, then $X_n \to_p X$. (The converse is not true.)

(iii) If $X_n \to_p X$, then $X_n \to_d X$. (The converse is not true.)

(iv) (Skorohod’s theorem). If $X_n \to_d X$, then there are $Y, Y_1, Y_2, \ldots$ defined on a common probability space such that $P_Y = P_X$, $P_{Y_n} = P_{X_n}$, $n = 1, 2, \ldots$, and $Y_n \to_{a.s.} Y$.

(v) If, for every $\epsilon > 0$, $\sum_{n=1}^\infty P(|X_n - X| \geq \epsilon) < \infty$, then $X_n \to_{a.s.} X$.

(A conditional converse of (i): $P(|X_n - X| \geq \epsilon)$ tends to 0 fast enough.)
Theorem 1.8 (continued)

(vi) If $X_n \to_p X$, then there is a subsequence $\{X_{n_j}, j = 1, 2, \ldots\}$ such that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$. (A partial converse of (i).)

(vii) If $X_n \to_d X$ and $P(X = c) = 1$, where $c \in \mathcal{R}$ is a constant, then $X_n \to_p c$. (A conditional converse of (i).)

(viii) Suppose that $X_n \to_d X$. Then, for any $r > 0$,

$$\lim_{n \to \infty} E|X_n|^r = E|X|^r < \infty$$

(1)

iff $\{|X_n|^r\}$ is uniformly integrable in the sense that

$$\lim_{t \to \infty} \sup_n E \left( |X_n|^r I_{\{|X_n| > t\}} \right) = 0.$$ 

(2)
Discussions on uniform integrability

► If there is only one random variable, then (2) is

$$\lim_{t \to \infty} E \left( |X|^r l_{\{|X|>t\}} \right) = 0,$$

which is equivalent to the integrability of $|X|^r$ (dominated convergence theorem).

► Sufficient conditions for uniform integrability:

$$\sup_n E|X_n|^{r+\delta} < \infty \quad \text{for a } \delta > 0$$

This is because

$$\lim_{t \to \infty} \sup_n E \left( |X_n|^r l_{\{|X_n|>t\}} \right) \leq \lim_{t \to \infty} \sup_n E \left( |X_n|^r l_{\{|X_n|>t\}} \frac{|X_n|^\delta}{t^\delta} \right)$$

$$\leq \lim_{t \to \infty} \frac{1}{t^\delta} \sup_n E \left( |X_n|^{r+\delta} \right)$$

$$= 0$$
Proof of Theorem 1.8

(i) The result follows from Lemma 1.4.

(ii) The result follows from Chebyshev’s inequality.

(iii) Let \( x \) be a continuity point of \( F_X \) and \( \epsilon > 0 \) be given. Then
\[
F_X(x - \epsilon) = P(X \leq x - \epsilon)
\leq P(X_n \leq x) + P(X \leq x - \epsilon, X_n > x)
\leq F_{X_n}(x) + P(|X_n - X| > \epsilon).
\]

Letting \( n \to \infty \), we obtain that
\[
F_X(x - \epsilon) \leq \liminf \limits_n F_{X_n}(x).
\]

Switching \( X_n \) and \( X \) in the previous argument, we can show that
\[
F_X(x + \epsilon) \geq \limsup \limits_n F_{X_n}(x).
\]

Since \( \epsilon \) is arbitrary and \( F_X \) is continuous at \( x \),
\[
F_X(x) = \lim \limits_{n \to \infty} F_{X_n}(x).
\]
Proof (continued)

(iv) The proof of this part can be found in Billingsley (1986, pp. 399-402).

(v) Let $A_n = \{ |X_n - X| \geq \epsilon \}$. The result follows from Lemma 1.4, Borel-Cantelli Lemma and continuity of probability measures.

(vi) $X_n \rightarrow_p X$ means $\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$ for every $\epsilon > 0$. That is, for every $\epsilon > 0$, $P(|X_n - X| > \epsilon) < \epsilon$ for $n > n_\epsilon$ ($n_\epsilon$ is an integer depending on $\epsilon$).

For every $j = 1, 2, \ldots$, there is a positive integer $n_j$ such that $P(|X_{n_j} - X| > 2^{-j}) < 2^{-j}$.

For any $\epsilon > 0$, there is a $k_\epsilon$ such that for $j \geq k_\epsilon$,

$P(|X_{n_j} - X| > \epsilon) < P(|X_{n_j} - X| > 2^{-j})$.

Since $\sum_{j=1}^{\infty} 2^{-j} = 1$, it follows from the result in (v) that $X_{n_j} \rightarrow_{a.s.} X$ as $j \rightarrow \infty$. 
Proof

(vii) The proof for this part is left as an exercise.

(viii) First, by part (iv), we may assume that $X_n \rightarrow_{a.s.} X$ (why?).
Proof

(vii) The proof for this part is left as an exercise.

(viii) First, by part (iv), we may assume that $X_n \rightarrow_{a.s.} X$ (why?).

Proof of (2) implies (1)

Note that (2) (the uniform integrability of $\{|X_n|^r\}$) implies that

$$\sup_n E|X_n|^r < \infty$$

By Fatou’s lemma, $E|X|^r \leq \lim \inf_n E|X_n|^r < \infty$.

Hence, (1) follows if we can show that

$$\limsup_n E|X_n|^r \leq E|X|^r. \quad (3)$$

For any $\epsilon > 0$ and $t > 0$, let $A_n = \{|X_n - X| \leq \epsilon\}$ and $B_n = \{|X_n| > t\}$. Then

$$E|X_n|^r = E(|X_n|^r I_{A_n^c \cap B_n}) + E(|X_n|^r I_{A_n^c \cap B_n^c}) + E(|X_n|^r I_{A_n})$$

$$\leq E(|X_n|^r I_{B_n}) + \epsilon^r P(A_n^c) + E|X_n I_{A_n}|^r.$$
Proof of (2) implies (1)

For $r \leq 1$, $|X_n I_{A_n}|^r \leq (|X_n - X|^r + |X|^r) I_{A_n}$ and

$$E |X_n I_{A_n}|^r \leq E[(|X_n - X|^r + |X|^r) I_{A_n}] \leq \epsilon^r + E|X|^r.$$ 

For $r > 1$, an application of Minkowski’s inequality leads to

$$E |X_n I_{A_n}|^r = E(|X_n - X) I_{A_n} + X I_{A_n}|^r$$

$$\leq \left\{ [E|(X_n - X) I_{A_n}|^r]^{1/r} + [E|X I_{A_n}|^r]^{1/r} \right\}^r$$

$$\leq \left\{ \epsilon + [E|X|^r]^{1/r} \right\}^r.$$ 

In any case, since $\epsilon$ is arbitrary, $\limsup_n E |X_n I_{A_n}|^r \leq E|X|^r$.

This result and the inequality in the end of the last page imply that

$$\limsup_n E |X_n|^r \leq \limsup_n E(|X_n|^r I_{B_n}) + t^r \lim_{n \to \infty} P(A_n^c) + E|X|^r,$$ 

Since $\lim_{n \to \infty} P(A_n^c) = 0$ and $\{|X_n|^r\}$ is uniformly integrable, letting $t \to \infty$ we obtain (3).
Proof of (1) implies (2)

Let \( \xi_n = |X_n|^r I_{B_n} - |X|^r I_{B_n}, B_n = \{|X_n| > t\}. \)
Then \( \xi_n \to a.s. \) 0 and \( |\xi_n| \leq t^r + |X|^r, \) which is integrable.
By the dominated convergence theorem, \( E\xi_n \to 0; \) this and (1) imply
\[
E(|X_n|^r I_{B_n}) - E(|X|^r I_{B_n}) \to 0.
\]
Since \( E|X|^r < \infty, \) by the dominated convergence theorem,
\[
\lim_{n \to \infty} E(|X|^r I_{\{|X_n-X|>t/2\}}) = 0.
\]
From the definition of \( B_n, \)
\[
|X|^r I_{B_n} \leq |X|^r I_{\{|X_n-X|>t/2\}} + |X|^r I_{\{|X|>t/2\}}.
\]
Hence
\[
\limsup_n E(|X_n|^r I_{B_n}) \leq \limsup_n E(|X|^r I_{B_n}) \leq E(|X|^r I_{\{|X|>t/2\}}).
\]
Letting \( t \to \infty, \) it follows from the dominated convergence theorem that
\[
\lim_{t \to \infty} \limsup_n E(|X_n|^r I_{B_n}) \leq \lim_{t \to \infty} E(|X|^r I_{\{|X|>t/2\}}) = 0.
\]
This proves (2).
Theorem 1.9 (useful sufficient and necessary conditions for convergence in distribution)

Let $X, X_1, X_2, \ldots$ be random $k$-vectors.

(i) $X_n \xrightarrow{d} X$ is equivalent to any one of the following conditions:

   (a) $E[h(X_n)] \xrightarrow{n \to \infty} E[h(X)]$ for every bounded continuous function $h$;
   (b) $\limsup_n P_{X_n}(C) \leq P_X(C)$ for any closed set $C \subset \mathbb{R}^k$;
   (c) $\liminf_n P_{X_n}(O) \geq P_X(O)$ for any open set $O \subset \mathbb{R}^k$.

(ii) (Lévy-Cramér continuity theorem). Let $\phi_X, \phi_{X_1}, \phi_{X_2}, \ldots$ be the ch.f.’s of $X, X_1, X_2, \ldots$, respectively.

   $X_n \xrightarrow{d} X$ iff $\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)$ for all $t \in \mathbb{R}^k$.

(iii) (Cramér-Wold device). $X_n \xrightarrow{d} X$ iff $c^\top X_n \xrightarrow{d} c^\top X$ for every $c \in \mathbb{R}^k$. 

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Proof of Theorem 1.9(i)

\( X_n \to_d X \) implies (a): By Theorem 1.8(iv) (Skorohod’s theorem), there exists a sequence of random vectors \( \{Y_n\} \) and a random vector \( Y \) such that \( P_{Y_n} = P_{X_n} \) for all \( n \), \( P_Y = P_X \) and \( Y_n \to_{a.s.} Y \). For bounded continuous \( h \), \( h(Y_n) \to_{a.s.} h(Y) \) and, by the dominated convergence theorem, \( E[h(Y_n)] \to E[h(Y)] \).

(a) follows from \( E[h(X_n)] = E[h(Y_n)] \) and \( E[h(X)] = E[h(Y)] \).
Proof of Theorem 1.9(i)

$x_n \rightarrow_d X$ implies (a): By Theorem 1.8(iv) (Skorohod’s theorem), there exists a sequence of random vectors $\{Y_n\}$ and a random vector $Y$ such that $P_{Y_n} = P_{X_n}$ for all $n$, $P_Y = P_X$ and $Y_n \rightarrow_{a.s.} Y$. For bounded continuous $h$, $h(Y_n) \rightarrow_{a.s.} h(Y)$ and, by the dominated convergence theorem, $E[h(Y_n)] \rightarrow E[h(Y)]$.

(a) follows from $E[h(X_n)] = E[h(Y_n)]$ and $E[h(X)] = E[h(Y)]$.

(a) implies (b): Let $C$ be a closed set and $f_C(x) = \inf\{\|x - y\| : y \in C\}$. Then $f_C$ is continuous.

For $j = 1, 2, \ldots$, define $\varphi_j(t) = l_{(-\infty, 0]} + (1 - jt)l_{[0, j^{-1}]}$.

Then $h_j(x) = \varphi_j(f_C(x))$ is continuous and bounded, $h_j \geq h_{j+1}$, $j = 1, 2, \ldots$, and $h_j(x) \rightarrow l_C(x)$ as $j \rightarrow \infty$.

Hence $\lim \sup_n P_{X_n}(C) \leq \lim_{n \rightarrow \infty} E[h_j(X_n)] = E[h_j(X)]$ for each $j$ (by (a)).

By the dominated convergence theorem, $E[h_j(X)] \rightarrow E[l_C(X)] = P_X(C)$. This proves (b).
Proof of Theorem 1.9(i) (continued)

For any open set $O$, $O^c$ is closed. Hence, (b) is equivalent to (c).

(b) and (c) imply $X_n \rightarrow d X$.

Let $x = (x_1, \ldots, x_k) \in R^k$ be a continuity point of $F_X$. Denote $(\ldots, x_1, \ldots) = (\ldots, x_1, \ldots) \times \cdots \times (\ldots, x_k, \ldots)$ and $(\ldots, x) = (\ldots, x_1, \ldots) \times \cdots \times (\ldots, x_k, \ldots)$.

From (b) and (c),

\[
F_X(x) = P_X((\ldots, x)) \leq \lim \inf_n P_X^n((\ldots, x)) \leq \lim \inf_n F_X^n(x) \leq \lim \sup_n F_X^n(x) = \lim \sup_n P_X^n((\ldots, x)) = F_X(x).
\]

This proves $X_n \rightarrow d X$.

Proof of Theorem 1.9(iii)

Note that $\phi c \tau X_n(u) = \phi X_n(uc)$ and $\phi c \tau X_n(u) = \phi X(uc)$ for any $u \in R$ and any $c \in R^k$. Hence, convergence of $\phi X_n$ to $\phi X$ is equivalent to convergence of $\phi c \tau X_n$ to $\phi c \tau X$ for every $c \in R^k$.

Then the result follows from part (ii).
Proof of Theorem 1.9(i) (continued)

For any open set $O$, $O^c$ is closed. Hence, (b) is equivalent to (c).

(b) and (c) imply $X_n \rightarrow_d X$:

Let $x = (x_1, \ldots, x_k) \in \mathcal{R}^k$ be a continuity point of $F_X$. Denote $(-\infty, x] = (-\infty, x_1] \times \cdots \times (-\infty, x_k]$ and $(-\infty, x) = (-\infty, x_1) \times \cdots \times (-\infty, x_k)$.

From (b) and (c),

$$F_X(x) = P_X((-\infty, x)) \leq \liminf_n P_{X_n}((-\infty, x)) \leq \liminf_n F_{X_n}(x)$$

$$\leq \limsup_n F_{X_n}(x) = \limsup_n P_{X_n}((-\infty, x]) \leq P_X((-\infty, x]) = F_X(x).$$

This proves $X_n \rightarrow_d X$. 

Proof of Theorem 1.9(i) (continued)

For any open set $O$, $O^c$ is closed. Hence, (b) is equivalent to (c).

(b) and (c) imply $X_n \rightarrow_d X$:
Let $x = (x_1, ..., x_k) \in \mathcal{R}^k$ be a continuity point of $F_X$. Denote

$(-\infty, x] = (-\infty, x_1] \times \cdots \times (-\infty, x_k]$ and

$(-\infty, x) = (-\infty, x_1) \times \cdots \times (-\infty, x_k)$.

From (b) and (c),

$$F_X(x) = P_X((-\infty, x)) \leq \lim_{n} \inf P_{X_n}((-\infty, x)) \leq \lim_{n} \inf F_{X_n}(x) \leq \lim_{n} \sup F_{X_n}(x) = \lim_{n} \sup P_{X_n}((-\infty, x]) \leq P_X((-\infty, x]) = F_X(x).$$

This proves $X_n \rightarrow_d X$.

Proof of Theorem 1.9(iii)

Note that $\phi_{c^\tau X_n}(u) = \phi_{X_n}(uc)$ and $\phi_{c^\tau X}(u) = \phi_X(uc)$ for any $u \in \mathcal{R}$ and any $c \in \mathcal{R}^k$. Hence, convergence of $\phi_{X_n}$ to $\phi_X$ is equivalent to convergence of $\phi_{c^\tau X_n}$ to $\phi_{c^\tau X}$ for every $c \in \mathcal{R}^k$. Then the result follows from part (ii).