1. Suppose, conditionally on \( M = m \) that \( Y \sim P(m) \), the Poisson distribution with parameter \( m \), and that \( M \) in turn has the gamma distribution \( M \sim G(\alpha \nu, \nu) \) with mean \( \mu = E(M) = \alpha \nu \) and coefficient of variation \( \sqrt{\nu} \). Show that the unconditional mean and variance are \( E(Y) = \mu \), and 
\[
Var(Y) = \alpha \nu + \alpha^2 \nu.
\]

Suppose now that \( Y \) has independent components generated in the above way with \( \mu_i = E(Y_i) \) not all equal. Show that if \( \nu_i = \nu \), a known constant, then the distribution of \( Y \) has the natural exponential-family form with variance function \( V(\mu) = \mu + \mu^2/\nu \), which is quadratic in \( \mu \). On the other hand if \( \alpha_i = \alpha \), a constant, show that the variance function has the standard over-dispersed Poisson form \( V(\mu) = \phi \mu \) with \( \phi = 1 + \alpha \), but that \( Y \) does not then have the linear exponential-family form.

More generally, if both \( \alpha \) and \( \nu \) vary according to the relations
\[
\alpha_i = \theta + \psi \mu_i, \quad \nu_i^{-1} = \psi + \theta \mu_i^{-1},
\]
show that \( V(\mu) = \mu + \theta \mu + \psi \mu^2 \) and that the distribution of \( Y \) again does not have the linear exponential-family form. Compare the exact likelihood with the corresponding quasi-likelihood in the second and third cases.

2. Consider the model
\[
Y_{1i} = \omega_1 + \rho R_i \cos \epsilon_i \cos \phi - \lambda R_i \sin \epsilon_i \sin \phi \\
Y_{2i} = \omega_2 + \rho R_i \cos \epsilon_i \sin \phi + \lambda R_i \sin \epsilon_i \cos \phi
\]
for an ellipse centered at \((\omega_1, \omega_2)\) with semi-axes of length \( \rho, \lambda \) inclined at an angle \( \phi \) to the \( x \)-axis. Assume that \( R_i \) are independent and identically distributed with mean 1, and independently of the \( \epsilon \)s. Construct an unbiased estimating function for the parameters \((\omega_1, \omega_2, \rho, \lambda, \phi)\).
Take as the elementary estimating functions
\[ R_i - 1 = \left( \frac{X_{i1}^2}{\rho^2} + \frac{X_{i2}^2}{\lambda^2} \right)^{1/2} - 1, \]
where
\[ X_{i1} = (Y_{i1} - \omega_1) \cos \phi + (Y_{i2} - \omega_2) \sin \phi = \rho R_i \cos \epsilon_i \]
\[ X_{i2} = -(Y_{i1} - \omega_1) \sin \phi + (Y_{i2} - \omega_2) \cos \phi = \lambda R_i \sin \epsilon_i. \]
Show that the required coefficients are
\[ D_{i1} = \cos \epsilon_i \cos \phi / \rho - \sin \epsilon_i \sin \phi / \lambda, \]
\[ D_{i2} = \cos \epsilon_i \sin \phi / \rho + \sin \epsilon_i \cos \phi / \lambda, \]
\[ D_{i3} = \cos^2 \epsilon_i / \rho, \]
\[ D_{i4} = \sin^2 \epsilon_i / \lambda, \]
\[ D_{i5} = (\rho - \lambda) \cos \epsilon_i \sin \epsilon_i. \]
Hence compute the information matrix for the five parameters.

3. Suppose that the random variables \( Y_1, \ldots, Y_n \) are independent with variance \( \text{var}(Y_i) = \sigma^2 \mu_i^2 \), where the coefficient of variation, \( \sigma \), is unknown. Suppose that inference is required for \( \beta_1 \), where
\[ \log(\mu_i) = \beta_0 + \beta_1 (x_i - \bar{x}_i)^2. \]
Show that the quasi-likelihood estimation of \( \beta_0, \beta_1 \) are uncorrelated with asymptotic variances
\[ \text{var}(\hat{\beta}_0) = \sigma^2 / n, \quad \text{var}(\hat{\beta}_1) = \sigma^2 / \sum_{i=1}^n (x_i - \bar{x}_i)^2. \]