ST3232: Design and Analysis of Experiments

Chen Zehua

Department of Statistics & Applied Probability

4:00-6:00 pm, Friday, February 19, 2016
Latin square design and its special nature

- A \( g \times g \) Latin square consists of three factors: two blocking factors and one treatment factor, each factor having \( g \) levels.
- The levels of the factors are arranged in the formation of a \( g \times g \) square.
- The levels of the two blocking factors are arranged as rows and columns of the square respectively.
- The treatment levels are arranged such that for any level it appears once and only once in each row and in each column.
The following is the formation of a $5 \times 5$ Latin square:

<table>
<thead>
<tr>
<th>Block F1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

In a latin square design, only one treatment level appears in each block; treatment levels are balanced at each level of the blocking factor.

This is to be compared with complete randomized block design: all $g$ levels of treatment appear in each block; treatment levels are balanced within each block.

Latin square design is particularly suitable in situations where different treatments cannot be administered simultaneously within each block.
Examples

- In agricultural experiments to compare fertilizers, to eliminating the effect of systematic gradients in fertility in the comparison, a field is divided into plots of rows and columns, fertilizers are applied to the plots according to the Latin square design.

- In animal experiments to compare diets, litters and birth order are used as blocking factors, diets as treatments.

- In clinical trials, individual patients and time are used as blocking factors, example of treatments could be dentures to be worn by the patients.

- A common nature of the above examples is that none of any treatments can be applied together with another one in a single block.
Randomization for Latin square design

- A random Latin square can be obtained from a standard square by permuting its rows and columns.
- A standard square is one in which the numerals 1, 2, \ldots, g are in numerical order in both the first row and the first column. For example:

```
1 2 3 4
2 1 4 3
3 4 1 2
4 3 2 1
```

- A particular standard square is the cyclic square:

```
1 2 3 4
2 3 4 1
3 4 1 2
4 1 2 3
```
Steps of randomization:

1. Choose at random a standard square, e.g., the first square above.
2. Permute the rows, e.g., arrange the rows by the permuted numerals (2, 4, 1, 3) to yield

\[
\begin{array}{cccc}
2 & 1 & 4 & 3 \\
4 & 3 & 2 & 1 \\
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
\end{array}
\]

3. Permute the columns, e.g., arrange the columns by the permuted numerals (4, 1, 3, 2) to yield

\[
\begin{array}{cccc}
3 & 2 & 4 & 1 \\
1 & 4 & 2 & 3 \\
4 & 1 & 3 & 2 \\
2 & 3 & 1 & 4 \\
\end{array}
\]
The randomization can be easily done by the following R-code:

```r
A = matrix(c(1,2,3,4, 2,1,4,3, 3,4,1,2, 4,3,2,1), byrow=T, ncol=4)
row = sample(4)
column = sample(4)
A[row, column]
```
Analysis of a single Latin square

Analysis of Variance

- In a single Latin square, three effects can be analyzed; that is, the main effects of the rows, columns and treatments.
- The ANOVA table is as follows:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>$g - 1$</td>
<td>$g \sum (\bar{y}_i - \bar{y})^2$</td>
<td>MSRow</td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td>$g - 1$</td>
<td>$g \sum (\bar{y}_j - \bar{y})^2$</td>
<td>MSCol</td>
<td></td>
</tr>
<tr>
<td>Treatment</td>
<td>$g - 1$</td>
<td>$g \sum (\bar{y}_k - \bar{y})^2$</td>
<td>MSTr</td>
<td>MSTr/MSE</td>
</tr>
<tr>
<td>Residual</td>
<td>$(g - 1)(g - 2)$</td>
<td>by subtraction</td>
<td>MSE</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$g^2 - 1$</td>
<td>$\sum \sum (y_{ijk} - \bar{y})^2$</td>
<td>MSE</td>
<td></td>
</tr>
</tbody>
</table>
Inference using regression model

The Latin square data can be represented by the following model:

\[ X = \mu_0 + \sum_{i=2}^{g} \alpha_i r_i + \sum_{j=2}^{g} \beta_j c_j + \sum_{k=2}^{g} \gamma_k t_k + \epsilon, \]

where \( r_i, c_j \) and \( t_j \) are dummy variables defined in the usual way as

\[
\begin{align*}
    r_i &= \begin{cases} 
        1, & \text{if row } i, \\
        0, & \text{otherwise, } i = 2, \ldots, g;
    \end{cases} \\
    c_j &= \begin{cases} 
        1, & \text{if column } j, \\
        0, & \text{otherwise, } j = 2, \ldots, g;
    \end{cases} \\
    t_k &= \begin{cases} 
        1, & \text{if treatment } k, \\
        0, & \text{otherwise, } k = 2, \ldots, g.
    \end{cases}
\end{align*}
\]

- The significance of treatment effect is tested by testing
  \[ H_0 : \gamma_2 = \cdots = \gamma_g = 0. \]
- Multiple comparison are based on the linear combinations of the \( \gamma_k \)'s.
Test for treatment effects

Let $\gamma = (\gamma_2, \ldots, \gamma_g)^\top$ and $\hat{\gamma}$ its LSE. The significance of treatment effect is tested by the $F$-ratio of the treatment in the ANOVA table.

The $F$-ratio can also be computed using Wald statistic as

$$F = \hat{\gamma}^\top \hat{\Sigma}^{-1}_\gamma \hat{\gamma} / (g - 1).$$

Under normality assumption, $F$ follows a $F$-distribution with df $g - 1$ and $N - 3(g - 1) - 1$, where $N$ is the total number of response measurements. If there is no missing value, $N = g^2$. 
Multiple comparison

- For a typical contrast \( C = \sum_{k=1}^{g} c_k \mu_k = \sum_{k=2}^{g} c_k \gamma_k \), the test statistic is computed as

\[
T = \frac{\sum_{k=1}^{g} c_k \bar{y}_k}{\sqrt{\text{MSE} \sum c_k^2 / g}},
\]

or

\[
T = \frac{\sum_{k=2}^{g} c_k \hat{\gamma}_k}{\sqrt{c^\tau \hat{\Sigma}_\gamma c}},
\]

where \( c = (c_2, \ldots, c_g)^\tau \).

- The two test statistics are the same.

- In the multiple comparison, the second df in Sheffe, Tukey, Dunnett’s criteria is given by \( N - 3(g - 1) - 1 \).
Example 11.1

The following table provides the data from a study comparing four formulas that were fed to newborn infants, each for one week. The response is the mean increase in weight recorded as ounces per day.

<table>
<thead>
<tr>
<th>Infant</th>
<th>1 (2)</th>
<th>2 (3)</th>
<th>3 (4)</th>
<th>4 (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.40</td>
<td>1.11</td>
<td>1.16</td>
<td>0.88</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>1.04</td>
<td>0.57</td>
<td>0.80</td>
</tr>
<tr>
<td>3</td>
<td>1.14</td>
<td>1.11</td>
<td>1.32</td>
<td>1.38</td>
</tr>
<tr>
<td>4</td>
<td>1.08</td>
<td>1.34</td>
<td>1.73</td>
<td>1.55</td>
</tr>
</tbody>
</table>

The ANOVA table for the above data is as follows:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infants</td>
<td>3</td>
<td>1.44</td>
<td>0.48</td>
<td></td>
</tr>
<tr>
<td>Weeks</td>
<td>3</td>
<td>0.64</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>Formulas</td>
<td>3</td>
<td>0.08</td>
<td>0.03</td>
<td>0.50</td>
</tr>
<tr>
<td>Residual</td>
<td>6</td>
<td>0.31</td>
<td>0.05</td>
<td></td>
</tr>
</tbody>
</table>
Example 11.1 (cont.)

- To test the significance of the treatment effect, the F-ratio of Formulas, $F = 0.50$, can be used. The $p$-value of the test is

$$P(F_{3,6} \geq 0.5) = 0.6959.$$ 

- The test statistic can also be computed using Wald statistic.
- As an illustration, the test statistic for the contrast $\mu_1 - \frac{1}{3}(\mu_2 + \mu_3 + \mu_4)$ is computed as follows.
- Computation using sample means: The four sample means are computed as 0.9825, 1.0100, 1.0450, 1.165, then

$$T = \frac{\sum_{k=1}^{g} c_k \bar{y}_{..k}}{\sqrt{\text{MSE} \sum c_k^2 / g}} = \frac{0.9825 - \frac{1}{3}(1.0100 + 1.0450 + 1.165)}{\sqrt{\frac{0.0521}{4}[1 + (-1/3)^2 + (-1/3)^2 + (-1/3)^2]}} = -0.6889.$$
Example 11.1 (cont.)

- Computation using regression coefficients:

\[
\hat{\gamma} = \begin{pmatrix} 0.0275 \\ 0.0625 \\ 0.1825 \end{pmatrix}, \quad \hat{\Sigma}_\gamma = \begin{pmatrix} 0.02607 & 0.01304 & 0.01304 \\ 0.01304 & 0.02607 & 0.01304 \\ 0.01304 & 0.01304 & 0.02607 \end{pmatrix}.
\]

\[
- \frac{1}{3} (\hat{\gamma}_2 + \hat{\gamma}_3 + \hat{\gamma}_4) = -\frac{1}{3} (0.0275 + 0.0625 + 0.1825) = -0.0908.
\]

\[
c^T \hat{\Sigma}_\gamma c
\]

\[
= (0.0275, 0.0625, 0.1825) \begin{pmatrix} 0.02607 & 0.01304 & 0.01304 \\ 0.01304 & 0.02607 & 0.01304 \\ 0.01304 & 0.01304 & 0.02607 \end{pmatrix} \begin{pmatrix} 0.0275 \\ 0.0625 \\ 0.1825 \end{pmatrix}
\]

\[
= 0.01738264.
\]

\[
T = \frac{\sum_{k=2}^{g} c_k \hat{\gamma}_k}{\sqrt{c^T \hat{\Sigma}_\gamma c}} = \frac{-0.09083333}{\sqrt{0.01738264}} = -0.6889495.
\]

- When there are missing values, computation should be based on the regression model.
A Remark

- The drawback of a single Latin square is that the residual degree of freedom is too small if $g$ is not large, which renders the power of the significance test low.
- Replicated Latin squares provide a remedy, which is to be discussed later.