Solutions to Midterm Exam

1. Consider the following process

\[(1 - 1.1B + 0.28B^2)\tilde{z}_t = (1 - 0.8B)a_t.\]

(i) State whether the process is stationary and/or invertible. Give reasons.

(ii) Obtain the first three \(\psi\) weights of the process.

(iii) Obtain the first three \(\pi\) weights of the process.

(iv) Obtain the autocorrelation coefficients \(\rho_1, \rho_2\) of the process.

Solution:

(i) The process is stationary since the roots of \(x^2 - 1.1x + 0.28 = 0\) are given by \(x_1 = 0.4, x_2 = 0.7\) which are within the unit circle. It is also invertible since the root of \(x - 0.8 = 0\) is 0.8 which is within the unit circle.

(ii)

\[
\psi_1 = \phi_1\psi_0 - \theta_1 = 1.1 - 0.8 = 0.3,
\psi_2 = \phi_1\psi_1 + \phi_2\psi_0 = 1.1(0.3) - 0.28 = 0.05,
\psi_3 = \phi_1\psi_2 + \phi_2\psi_1 = 1.1(0.05) - 0.28(0.3) = -0.029,
\]

(iii)

\[
\begin{align*}
\pi_1 &= \theta_1\pi_0 + \phi_1 = 0.8(-1) + 1.1 = 0.3, \\
\pi_2 &= \theta_1\pi_1 + \phi_2 = 0.8(0.3) - 0.28 = -0.04, \\
\pi_3 &= \theta_1\pi_2 = 0.8(-0.04) = -0.032
\end{align*}
\]

(iv) The variance and the first two autocovariances of the process can be solved from the following equations:

\[
\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 - \sigma_a^2(\theta_0\psi_0 + \theta_1\psi_1),
\]
\[ \gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_2 - \sigma_a^2 \theta_1 \psi_0, \]
\[ \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0. \]

**Note:** \( \phi_1 = 1.1, \phi_2 = -0.28, \theta_1 = 0.8, \phi_1 = 0.3 \)

and \( \theta_0 = -1, \psi_0 = 1. \)

**Hence**
\[ \gamma_0 = 1.1 \gamma_1 - 0.28 \gamma_2 + 0.76 \sigma_a^2, \]
\[ \gamma_1 = 1.1 \gamma_0 - 0.28 \gamma_2 - 0.8 \sigma_a^2, \]
\[ \gamma_2 = 1.1 \gamma_1 - 0.28 \gamma_0. \]

Solving the above equations yields:
\[ \gamma_0 = 1.099636 \sigma_a^2, \gamma_1 = 0.32 \sigma_a^2. \]

**Hence**
\[ \rho_1 = \gamma_1 / \gamma_0 = 0.291, \]
\[ \rho_2 = 1.1 \rho_1 - 0.28 \rho_0 = 1.1(0.291) - 0.28 = 0.0401. \]

2. For the following process
\[ (1 - 0.6B) \nabla z_t = (1 - 0.3B) a_t, \]

(i) write the process in inverted form;

(ii) write the process in random shock form;

(iii) write the process as a complementary function plus a particular integral in relation to an origin \( k \) (\( < t \)). [for the complementary function, only give the form of the function. You don’t need to compute the values of the coefficients in the function.]

**Solution**

(i) Since \( \varphi(B) = (1 - 0.6B)(1 - B) = 1 - 1.6B + 0.6B^2 \), we can identify \( \varphi_1 = 1.6, \varphi_2 = -0.6. \)

For \( j = 1, 2, \)
\[ \pi_1 = \theta_1 \pi_0 + \varphi_1 = 0.3(-1) + 1.6 = 1.3 \]
\[ \pi_2 = \theta_1 \pi_1 + \varphi_2 = 0.3(1.3) - 0.6 = -0.21. \]
For $j \geq 3$,
\[
\pi_j = \theta_1 \pi_{j-1} = \pi_2 \theta_1^{j-2} = -0.21(0.3)^{j-2}.
\]
The inverted form is then given by
\[
z_t = 1.3z_{t-1} - 0.21 \sum_{j=2}^{\infty} 0.3^{j-2}z_{t-j} + a_t.
\]

(ii) We have
\[
\begin{align*}
\psi_1 &= \phi_1 \psi_0 - \theta_1 = 1.6(1) - 0.3 = 1.3, \\
\psi_2 &= \phi_1 \psi_1 + \phi_2 \psi_0 = 1.6(1.3) + (-0.6)(1) = 1.48.
\end{align*}
\]
For $j \geq 2$, $\psi_j$ satisfy
\[
\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} = 0.
\]
The general solution for $\psi_j$ is given by
\[
\psi_j = b_0 + b_1 0.6^j,
\]
which is valid for $j \geq 0$. Setting $j = 0$ and 1,
\[
\begin{align*}
b_0 + b_1 &= 1, \\
b_0 + 0.6b_1 &= 1.3.
\end{align*}
\]
Solving the above equation yields
\[
b_0 = 1.75, \ b_1 = -0.75.
\]
Hence the random shock form is given by
\[
z_t = a_t + \sum_{j=1}^{\infty} [1.75 - 0.75(0.6)^j]a_{t-j}.
\]

(iii) The form is given by
\[
z_t = C_k(t-k) + I_k(t-k),
\]
where

\[ I_k(t - k) = a_t + \psi_1 a_{t-1} + \cdots + \psi_{t-k-1} a_{k+1} = a_t + \sum_{j=1}^{t-k-1} [1.75 - 0.75(0.6)^j] a_{t-j}. \]

The complementary function \( C_k(t - k) \) has the general form:

\[ C_k(t - k) = b_0^{(k)} + b_1^{(k)} 0.6^{t-k}. \]

3. The following observations represent the values \( z_{96}, \ldots, z_{100} \) from a series fitted by the model \((1 - 0.3B)\nabla z_t = (1 - 0.4B)a_t:\)

\[ 19.21, 20.18, 20.12, 18.69, 18.23. \]

Given that \( a_{97} = 0.9. \)

(i) Generate the forecasts \( \hat{z}_{100}(l) \) for \( l = 1, 2, 3. \)

(ii) With \( \hat{\sigma}_a^2 = 1, \) calculate the variances \( V(l) \) of the forecast errors.

**Solution**

(i) Using the recursive formula

\[ a_t = z_t - 1.3z_{t-1} + 0.3z_{t-2} + 0.4a_{t-1}, \]

we obtain

\[ a_{98} = 0.009, \ a_{99} = -1.4084, \ a_{100} = -0.59436. \]

Then, we have

\[ \hat{z}_{100}(1) = 1.3z_{100} - 0.3z_{99} - 0.4a_{100} = 18.32974, \]
\[ \hat{z}_{100}(2) = 1.3\hat{z}_{100}(1) - 0.3z_{100} = 18.35967, \]
\[ \hat{z}_{100}(3) = 1.3\hat{z}_{100}(2) - 0.3\hat{z}_{100}(1) = 18.36864. \]
(ii) The variances are given by the formula

\[ V(l) = \sigma_a^2 (1 + \sum_{j=1}^{l-1} \psi_j^2). \]

we have

\[
\begin{align*}
\psi_1 &= 1.3\psi_0 - 0.4 = 0.9, \\
\psi_2 &= 1.3\psi_1 - 0.3\psi_0 = 1.3 \times 9 - 0.3 = 0.87.
\end{align*}
\]

Hence

\[
\begin{align*}
V(1) &= \sigma_a^2 = 1, \\
V(2) &= \sigma_a^2 (1 + \psi_1^2) = 1 + 0.9^2 = 1.81, \\
V(3) &= \sigma_a^2 (1 + \psi_1^2 + \psi_2^2) = 1.81 + 0.87^2 = 2.5669.
\end{align*}
\]