CHAPTER 2: Markov Chains (part 2)

Basic questions

First step analysis

1 Some Markov Chain Models

1.1 An inventory Model

Let X_n denote the number consumable items somebody has at the end of a week. In week n, ξ_n items will be consumed with

$$P(\xi_n = k) = a_k, \quad k = 0, 1, 2, ...$$

on the weekend, if $X_n > s$, no buying; if $X_n < s$, $S - X_n$ items will be bought. Then

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1}, & \text{if } s < X_n \le S \\ S - \xi_{n+1}, & \text{if } X_n \le s. \end{cases}$$

1. It is clear that $\{X_n : n = 0, 1, 2, ...\}$ is a MC (why?)

2. Let $s = 0$, $S = 2$ with $P(\xi_n = 0) = 0.5$, $P(\xi_n = 1) = 0.4$ and $P(\xi_n = 2) = 0.1$, what is the transition probability matrix ?

$$\mathbf{P} = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{cccc} 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0.1 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 & 0 \\ 0 & 0.1 & 0.4 & 0.5 \end{array} \right\| \end{array}$$

('-' means 'borrowing')

3. (unsolved problem) how frequently you need to purchase?

$$\mathbf{P}^{\infty} = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{cccc} 0.0444 & 0.2333 & 0.4444 & 0.2778 \\ 0.0444 & 0.2333 & 0.4444 & 0.2778 \\ 0.0444 & 0.2333 & 0.4444 & 0.2778 \\ 0.0444 & 0.2333 & 0.4444 & 0.2778 \end{array} \right\| \end{array}$$

we will discuss it later. Note that P^n converges as $n \to \infty$

1.2 The Ehrenfest Urn Models

suppose there are $2a$ balls. amongst them, k balls are in container (urn) A, and $2a-k$ are in container B. A balls is selected at random (all selections are equally likely) from the totally $2a$ balls and moved to the other container. Let Y_n be the number of balls in container A at the nth stage. Then Y_n is a Markov Chain with

$$S = \{0, 1, 2, \cdots, 2a\}.$$

and

$$\begin{aligned} P(Y_{n+1} = i+1 | Y_n = i) &= \frac{2a-i}{2a}, \text{ if } \ 0 \le i \le 2a \\ P(Y_{n+1} = i-1 | Y_n = i) &= \frac{i}{2a}, \\ P(Y_{n+1} = j | Y_n = i) &= 0, \text{ if } \ j \ne i-1, i+1. \end{aligned}$$

For $a = 2$, the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Vmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$$

We have

$$\mathbf{P^{2k}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Vmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.125 & 0 & 0.75 & 0 & 0.125 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.125 & 0 & 0.75 & 0 & 0.125 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.125 & 0 & 0.75 & 0 & 0.125 \end{Vmatrix}$$

and

$$\mathbf{P^{2k+1}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Vmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.125 & 0 & 0.75 & 0 & 0.125 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.125 & 0 & 0.75 & 0 & 0.125 \\ 0 & 0.5 & 0 & 0.5 & 0 \end{Vmatrix}$$

Note that P^n does not converge.

1.3 A discrete queueing Markov Chain

Customers arrive for service and take their place in a waiting line[1]. Suppose that

$$P(k \text{ customer arrive in a service period })$$

[1]A more advanced topic

$$= P(\xi_n = k)$$
$$= a_k.$$

where ξ_n has the same distribution as ξ. In each service period, only one customer is served. Let X_n be the customers waiting for service. Then

$$X_{n+1} = \max(X_n - 1, 0) + \xi_n.$$

Based on this, the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{array} \right\| \end{array}$$

Unsolved questions

1. what is $\mathbf{P}^n$?
2. Intuitively,
   (a) If $E\xi > 1$, then the number of customers waiting for service will increase infinitely.
   (b) If $E\xi < 1$, what is the probability that there will be k customers waiting for service. [if you are the only hairdresser in a barbershop, how many chairs you need to provide?]

2 First Step Analysis

2.1 Motivating Example

In the Gambler's Ruin example (with <u>N=4</u> and $X_0 = 3$),

1. what is probability that the gambler eventually goes broke (or win N$)?
2. On average, how many games he can play before he goes broke?
3. how many times he can have $0 < k \le N$ dollars before the game ends?

Recall (for N = 4) that

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

States 0 and 4 are absorbing states.

Definition If $P_{ii} = 1$, then state i is an absorbing state.

Two important variables

$$T = \min_n \{n : X_n = 0 \text{ or } X_n = N\}\text{—no. of steps before the end}$$
$$\{X_T = 0\}\text{—}\{\text{the game ends in state } 0\}$$
$$(\textit{or } \{X_T = N\}\text{—}\{\text{the game ends in state } 0\}.)$$

By this notation, we go back to the questions

* Question 1 is to calculate $u_3 = P(X_T = 0|X_0 = 3)$ More generally,

$$u_i = P(X_T = 0|X_0 = i),$$

where $i = 1, 2, 3$

* Question 2 is to calculate $v_3 = E(T|X_0 = 3)$ More generally,

$$v_i = E(T|X_0 = i), \quad i = 1, 2, 3$$

* Question 3 is to calculate $w_{3k} = E(\sum_{n=0}^{T-1} I(X_n = k)|X_0 = 3)$ More generally,

$$w_{ik} = E(\sum_{n=0}^{T-1} I(X_n = k)|X_0 = i), i = 1, 2, 3$$

Start from step 1, define

$$T' = \min_n \{n - 1 : X_n = 0 \text{ or } X_n = N\}\text{—no. of steps before the end start with 1}$$

If i is not a absorbing state, then given $X_0 = i$ and $X_1 = j$

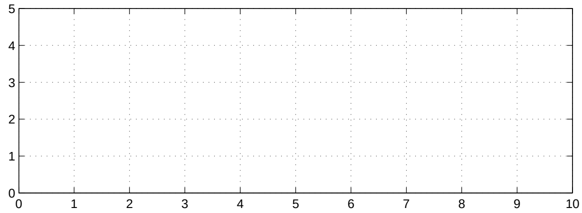
$$T = 1 + T'$$

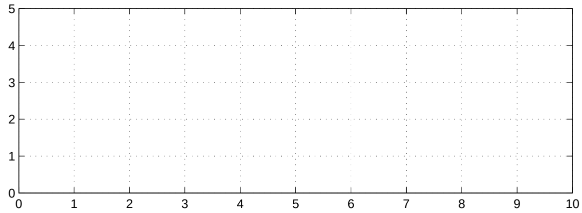
and given $X_0 = i$, T has the same statistical distribution as T' given $X_1 = i$.

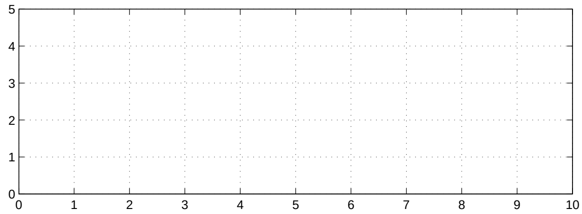
[*Proof: Recall that*

$$P(X_{n_1} = i_{n_1}, ..., X_{n_k} = i_{n_k}|X_0 = i) = P(X_{n_1+m} = i_{n_1}, ..., X_{n_k+m} = i_{n_k}|X_m = i)$$

for any $n_1, ..., n_k > 0$. *Thus the above claim follows.*]







If state i is not a absorbing state, we have the following *useful* relations

$$\begin{aligned}
P(X_T = 0|X_0 = i, X_1 = j) &= P(X_{T'+1} = 0|X_0 = i, X_1 = j) \\
&= P(X_{T'+1} = 0|X_1 = j) \quad \text{(by MC property)} \\
&= P(X_T = 0|X_0 = j) \quad \text{(by the relationship between } T' \text{ and } T) \\
&= u_j
\end{aligned}$$

$$\begin{aligned}
E(T|X_0 = i, X_1 = j) &= 1 + E(T'|X_0 = i, X_1 = j) \\
&= 1 + E(T'|X_1 = j) \quad \text{(by MC property)} \\
&= 1 + E(T|X_0 = j) \quad \text{(by the relationship between } T' \text{ and } T) \\
&= 1 + v_j
\end{aligned}$$

and

$$\begin{aligned}
&E(\sum_{n=0}^{T-1} I(X_n = k)|X_0 = i, X_1 = j) \\
&= \begin{cases} 1 + E(\sum_{n=1}^{T-1} I(X_n = k)|X_0 = i, X_1 = j) & \text{if } \ i = k \\ \\ 0 + E(\sum_{n=1}^{T-1} I(X_n = k)|X_0 = i, X_1 = j) & \text{if } \ i \neq k \end{cases}
\end{aligned}$$

$$= \begin{cases} 1 + E(\sum_{n=1}^{T-1} I(X_n = k)|X_1 = j) & \text{if } \ i = k \\ 0 + E(\sum_{n=1}^{T-1} I(X_n = k)|X_1 = j) & \text{if } \ i \neq k \end{cases}$$

$$= \begin{cases} 1 + E(\sum_{n=1}^{T'+1-1} I(X_n = k)|X_1 = j) & \text{if } \ i = k \\ 0 + E(\sum_{n=1}^{T'+1-1} I(X_n = k)|X_1 = j) & \text{if } \ i \neq k \end{cases}$$

$$= \begin{cases} 1 + E(\sum_{n=0}^{T-1} I(X_n = k)|X_0 = j) & \text{if } \ i = k \\ 0 + E(\sum_{n=0}^{T-1} I(X_n = k)|X_0 = j) & \text{if } \ i \neq k \end{cases}$$

$$= \begin{cases} 1 + w_{jk} & \text{if } \ i = k \\ 0 + w_{jk} & \text{if } \ i \neq k \end{cases}$$

Calculation of u_i in the gambler's example

Let $u_i = P(X_T = 0|X_0 = i), i = 0, 1, 2, 3, 4$. It is easy to see from the example (N=4)

$$u_0 = P(X_T = 0|X_0 = 0) = 1, \quad u_4 = P(X_T = 4|X_0 = 0) = 0$$

By the important relations, we have

$$\begin{aligned} u_3 &= P(X_T = 0|X_0 = 3) \\ &= \sum_{i=0}^{4} P(X_T = 0|X_0 = 3, X_1 = i)P(X_1 = i|X_0 = 3) \\ &= \sum_{i=0}^{4} P(X_T = 0|X_0 = i)P(X_1 = i|X_0 = 3) \\ &= \sum_{i=0}^{4} u_i P(X_1 = i|X_0 = 3) \\ &= \sum_{i=0}^{4} u_i p_{3i} \end{aligned}$$

More generally,

$$u_j = \sum_{i=0}^{4} u_i p_{ji}, \quad j = 1, 2, 3$$

i.e.

$$\begin{aligned} u_0 &= 1, \\ u_1 &= p_{10}u_0 + p_{11}u_1 + p_{12}u_2 + p_{13}u_3 + p_{14}u_4 \end{aligned}$$

$$\begin{aligned}
u_2 &= p_{20}u_0 + p_{21}u_1 + p_{22}u_2 + p_{23}u_3 + p_{24}u_4 \\
u_3 &= p_{30}u_0 + p_{31}u_1 + p_{32}u_2 + p_{33}u_3 + p_{34}u_4 \\
u_4 &= 0
\end{aligned}$$

or

$$\begin{aligned}
u_0 &= 1 \\
u_1 &= \frac{2}{3}u_0 + 0u_1 + \frac{1}{3}u_2 + 0u_3 + 0u_4 \\
u_2 &= 0u_0 + \frac{2}{3}u_1 + 0u_2 + \frac{1}{3}u_3 + 0u_4 \\
u_3 &= 0u_0 + 0u_1 + \frac{2}{3}u_2 + 0u_3 + \frac{1}{3}u_4 \\
u_4 &= 0
\end{aligned}$$

we have

$$u_1 = \frac{14}{15}, \quad u_2 = \frac{4}{5}, \quad u_3 = \frac{8}{15}$$

Interestingly,

$$\mathbf{P}^n \to \left\| \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 14/15 & 0 & 0 & 0 & 1/15 \\ 4/5 & 0 & 0 & 0 & 1/5 \\ 8/15 & 0 & 0 & 0 & 7/15 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right\|$$

Calculation of $v_i = E(T|X_0 = i)$ in the gambler's example

Note that

$$v_0 = E(T|X_0 = 0) = 0, \quad v_4 = E(T|X_0 = 4) = 0$$

By the important relations

$$\begin{aligned}
v_3 &= E(T|X_0 = 3) \\
&= \sum_{i=0}^{4} E(T|X_0 = 3, X_1 = i)E(X_1 = i|X_0 = 3) \\
&= \sum_{i=0}^{4} (1 + v_i) \times P(X_1 = i|X_0 = 3) \\
&= \sum_{i=0}^{4} p_{3i} + \sum_{i=0}^{4} v_i p_{3i}
\end{aligned}$$

Generally

$$v_j = E(T|X_0 = j) = 1 + \sum_{i=0}^{4} v_i p_{ji}, \;\; j = 1, 2, 3.$$

or

$$\begin{aligned}
v_0 &= 0 \\
v_1 &= 1 + \frac{2}{3}v_0 + 0v_1 + \frac{1}{3}v_2 + 0v_3 + 0v_4 \\
v_2 &= 1 + 0v_0 + \frac{2}{3}v_1 + 0v_2 + \frac{1}{3}v_3 + 0v_4 \\
v_3 &= 1 + 0v_0 + 0v_1 + \frac{2}{3}v_2 + 0v_3 + \frac{1}{3}v_4 \\
v_4 &= 0
\end{aligned}$$

we have

$$v_1 = \frac{11}{5}, \quad u_2 = \frac{18}{5}, \quad v_3 = \frac{17}{5}$$

Calculation of $w_{i3} = E(\sum_{n=0}^{T-1} I(X_n = 3)|X_0 = i)$ in the gambler's example

Following the same idea

$$\begin{aligned}
w_{33} &= \sum_{j=0}^{3} E(\sum_{n=0}^{T-1} I(X_n = 3)|X_0 = 3, X_1 = j)p_{3j} \\
&= \sum_{j=0}^{3}\{1 + E(\sum_{n=0}^{T-1} I(X_n = 3)|X_0 = j)\}p_{3j} \\
&= \sum_{j=0}^{3} p_{3j} + \sum_{j=0}^{3} E(\sum_{n=0}^{T-1} I(X_n = 3)|X_0 = j)p_{3j} \\
&= 1 + p_{30}w_{03} + p_{31}w_{13} + p_{32}w_{23} + p_{33}w_{33} + p_{34}w_{43}
\end{aligned}$$

and similarly for $i \neq 3$

$$w_{i3} = p_{i0}w_{03} + p_{i1}w_{13} + p_{i2}w_{23} + p_{i3}w_{33} + p_{i4}w_{43}$$

special cases

$$w_{03} = 0, w_{43} = 0$$

Combining these, we have

$$\begin{aligned}
w_{03} &= 0 \\
w_{13} &= \frac{2}{3}w_{03} + 0w_{13} + \frac{1}{3}w_{23} + 0w_{33} + 0w_{43} \\
w_{23} &= 0w_{03} + \frac{2}{3}w_{13} + 0w_{23} + \frac{1}{3}w_{33} + 0w_{43} \\
w_{33} &= 1 + 0w_{03} + 0w_{13} + \frac{2}{3}w_{23} + 0w_{33} + \frac{1}{3}w_{43} \\
w_{43} &= 0
\end{aligned}$$

we have

$$w_{13} = 0.2, \quad w_{23} = 0.6, \quad w_{33} = 1.4.$$

Another example

Example A Markov chain $X_t : t = 0, 1, ...$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that when the process moves to state 2, it does so from state 1?

[Solution: Let $T = \min\{n \geq 0 : X_n = 2\}$ and

$$z_i = P\{X_{T-1} = 1 | X_0 = i\}, \text{ for } \ i = 0, 1$$

then we have

$$\begin{aligned} z_0 &= P\{X_{T-1} = 1 | X_0 = 0, X_1 = 0\} p_{00} \\ &\quad + P\{X_{T-1} = 1 | X_0 = 0, X_1 = 1\} p_{01} \\ &\quad + P\{X_{T-1} = 1 | X_0 = 0, X_1 = 2\} p_{02} \\ &= P\{X_{T-1} = 1 | X_0 = 0\} p_{00} + P\{X_{T-1} = 1 | X_0 = 1\} p_{01} + 0 \\ &= z_0 p_{00} + z_1 p_{01} \end{aligned}$$

i.e.

$$z_0 = 0.3 z_0 + 0.2 z_1 \tag{2.1}$$

$$\begin{aligned} z_1 &= P\{X_{T-1} = 1 | X_0 = 1, X_1 = 0\} p_{10} \\ &\quad + P\{X_{T-1} = 1 | X_0 = 1, X_1 = 1\} p_{11} \\ &\quad + P\{X_{T-1} = 1 | X_0 = 1, X_1 = 2\} p_{12} \\ &= P\{X_{T-1} = 1 | X_0 = 0\} p_{00} + P\{X_{T-1} = 1 | X_0 = 1\} p_{01} + 0 \\ &= z_0 p_{10} + z_1 p_{11} + 1 * p_{12} \end{aligned}$$

i.e.

$$z_1 = 0.5 z_0 + 0.1 z_1 + 0.4 \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$z_0 = 0.1509 \quad z_1 = 0.5283$$