Chapter 2: Markov Chains (part 1)

Basic questions

1. Stochastic process; 2. properties of a Markov chain; 3. One step transition probability (matrix); 4. multi-step transition probability (matrix); 5. State-diagram associated with transition probabilities.

1 Stochastic Process

Definition: A stochastic Process $\{X_t, t \in T\}$ is a collection of random variables where t is the time index (or space index as in spatial analysis).

T is the **index set**

The set of possible values of X_t is **state space**, denoted by S.

If $X_t = i \in S$, then we say that the process is in state i at time t.

X_0 is usually called the initial state.

For convenience, sometimes we write $X(t)$ for X_t.

Example Toss a coin n times. Let X_i be the outcome of the ith toss. Then $\{X_t : t = 0, 1, 2, \cdots n\}$ is a stochastic Process with

$$T = \{0, 1, 2, \cdots, n\} \qquad S = \{H, T\}.$$

Example (Gambler's ruin) A gambler starts with 3\$, win 1\$ with probability 1/3; losses 1\$ with probability 2/3. He must leave when he goes broke or he wins N\$. Let X_t be the money he has after the t'th game. Then $\{X_t : t = 0, 1, 2, \cdots\}$ is a stochastic Process with

$$T = \{0, 1, 2, \cdots, \} \qquad S = \{0, 1, \cdots, N\}.$$

Example (Poisson Process) X_t counts the number of times that a specified event occurs during the time period from 0 to t. Then $\{X_t : t \in [0, \infty]\}$ is a stochastic process with

$$T = [0, \infty], \qquad S = \{0, 1, 2, \cdots, \}.$$

Example (Stock market index) X_t is the S&P 500 index in the t'th day of this year. Then $\{X_t : t = 1, 2, \cdots, 365\}$ is a stochastic process with

$$T = \{1, 2, \cdots, 365\}, \qquad S = (0, \infty).$$

Classification of a stochastic process :

If T is a countable set — $\{X_t : t \in T\}$ a discrete time stochastic process .

If T is a continuum — $\{X_t : t \in T\}$ a continuous-time stochastic process .

If S is a countable set — $\{X_t : t \in T\}$ a discrete-state stochastic process .

If S is a continuum — $\{X_t : t \in T\}$ a real-state stochastic process .

Example In examples above, which is discrete time stochastic process , which is continuous time stochastic process , ...

A stochastic process is defined by T, S and the joint distribution at any finite number of time points

2 Definition of Markov Chain

Definition: **Markov Chain**[1] Let $\{X_t : t \in T\}$ be a stochastic process with discrete-state space S and discrete-time space T satisfying

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

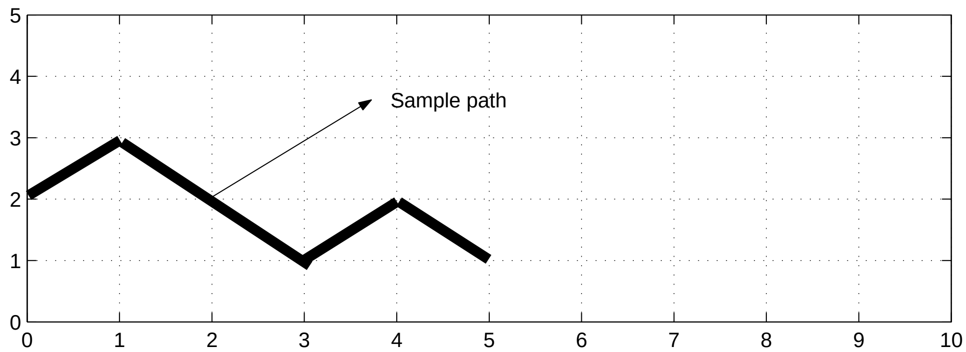
for any set of state $i_0,\ i_1, \cdots, i_{n-1}, i, j$ in S and $n \geq 0$ is called a **Markov Chain (MC)**.

[1]Markov chain was named after A.A. Markov who developed the theoretical functions for the finite state Markov chains in the 1900s. An interesting example from the 19th century is called the Galton-Watson process which attempts to answer the question of when and with what probability would a given family name become extinct.

A process with the property stated in the above definition is said to have the **Markovian property**, i.e. the conditional distribution of any future state X_{n+1} depends on the present state and is independent of the past states.

Example (Gambler's ruin (continued) with N = 5)

$$
\begin{aligned}
& P(X_{t+1} = j | X_t = i) \\
= & \begin{cases} 1/3, & j = i+1 \text{ and } 5 > i > 0; \\ 2/3, & j = i-1 \text{ and } \quad 5 > i > 0; \\ 1, & \text{if } j = i = 0; \\ 1, & \text{if } j = i = 5; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$



Definition Let $P_{ij}^{n,n+1} = P(X_{n+1} = j | X_n = i)$, called **one-step transition probability**.

It is easy to see that the one-step transition probability dependents on n and i, j.

Definition: (**Stationary transition probability**[2]) A Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$ with state space S is said to have stationary transition probability if

$$
\begin{aligned}
& P(X_{n+1} = j | X_n = i) \\
= & P(X_n = j | X_{n-1} = i) \\
= & \cdots \\
= & P(X_1 = j | X_0 = i)
\end{aligned}
$$

for each $i, j \in S$, i.e. probability of the one-step transition does not change as n in creases.

A counterexample of stationary Markov Chain: for people's promotion, time space is T: people's age; State space $\{junior, senior\}$, then

...

[2] In this module, we consider Markov chains with stationary transition probability only.

$> P(X_{50} = senior | X_{49} = junior)$

$> ...$

$> P(X_{40} = senior | X_{39} = junior)$

$> P(X_{39} = senior | X_{38} = junior)$

...

One-step transition probability matrix We can use a matrix to denote the transition probability.

$$\mathbf{P} = (p_{ij}) = \begin{matrix} & 0 & 1 & 2 & \cdots \\ 0 & p_{00} & p_{01} & p_{02} & \cdots \\ 1 & p_{10} & p_{11} & p_{12} & \cdots \\ 2 & p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \cdots & & & \end{matrix}$$

Example Toss a coin (probability of Head 0.6) independently n times. X_i is outcome of i'th toss. Then the one-step transition matrix is

$$\mathbf{P} = (p_{ij}) = \begin{matrix} & H & T \\ H & 0.6 & 0.4 \\ T & 0.6 & 0.4 \end{matrix}$$

Example (Gambler's ruin (continued) with N = 5)

$$P(X_{t+1} = j | X_t = i) = \begin{cases} 1/3, & j = i+1 \text{ and } 5 > i > 0; \\ 2/3, & j = i-1 \text{ and } 5 > i > 0; \\ 1, & \text{if } j = i = 0; \\ 1, & \text{if } j = i = 5; \\ 0, & \text{otherwise} \end{cases}$$

Then the one-step transition matrix is

$$\mathbf{P} = (p_{ij}) = \begin{matrix} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 2 & 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 3 & 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 4 & 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Properties of P

$$(1) \qquad p_{ij} \geq 0, \quad \text{for} \quad i, j \in S$$

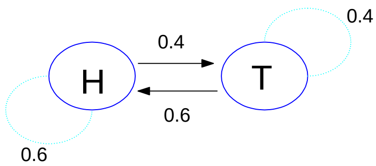
$$(2) \qquad \sum_{j=0}^{\infty} p_{ij} = 1, \quad \text{for} \quad i \in S$$

——summation of each row in $P$is 1.

Proof: It follows immediately from the definition of probability and law of total probability.

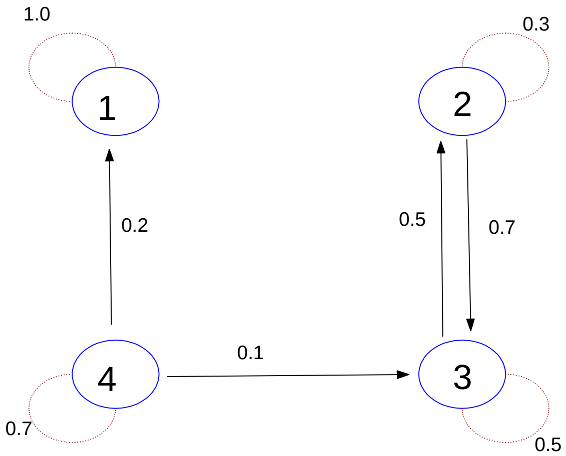
State-diagram associated with transition probabilities A diagram for the Markov chain that represents the movement and the probability for the movement.

Example In the example of tossing a coin



Example Draw a state diagram associated with the following transition probability matrix.

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{cccc} 1.0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.2 & 0 & 0.1 & 0.7 \end{array} \right\| \end{array}$$



Example (Gambler's ruin (continued)) Draw a state diagram associated with the following transition probability matrix.

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

States 0 and 5 are called **absorbing states** since once these states are reached, the process can never leave.

Example A salesman lives in Town A and is responsible for the sales consisting of Towns A, B and C. Each week he is required to visit a different town. When he is in his own town, it makes no difference which town he visits next so he slips a coin and if it is heads he goes to B and if tails he goes to C. However, after spending a week away from home, he has a slight preference for going home so when he is in either towns B or C, he has a slight preference for going home so when he is in either towns B or C, he flips two coins. If two heads occur, then he goes to the other town; otherwise he goes to A.

The successive towns that the salesman visits form a Markov chain with state space $\{A, B, C\}$ where random variable X_n equals A, B or C according to his location during week n. The transition probability matrix of this problem is given by

$$\mathbf{P} = \begin{Vmatrix} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{Vmatrix}$$

Given a Markov chain, what do we want to know next?

- After t steps, what's the probability that the MC is in state i?
- As $t \to \infty$, what's the probability that the MC is in state i?
- Given that we've taken t steps, what's the probability we've ever been in state i?
- What's the expected number of steps before we reach state i for the first time?

3 Chapman-Kolmogorov Equations

Denote the **n-step transition probability** of the Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$ with state space S by

$$p_{ij}^{(n)} = P(X_{m+n} = j | X_m = i),$$
$$\forall m, \ n = 0, 1, 2, \cdots, \quad \forall i, j \in S.$$

Similarly, we define the n-step transition probability matrix as

$$\mathbf{P}^{(n)} = (p_{ij}^{(n)}) = \left\| \begin{array}{cccc} p_{00}^{(n)} & p_{01}^{(n)} & p_{02}^{(n)} & \cdots \\ p_{10}^{(n)} & p_{11}^{(n)} & p_{12}^{(n)} & \cdots \\ \vdots & & & \\ p_{i0}^{(n)} & p_{i1}^{(n)} & p_{i2}^{(n)} & \cdots \\ \vdots & & & \end{array} \right\| .$$

Example Consider the salesman example where the salesman starts from Town B. The transition probability matrix indicates that the probability of being in Town A after one step (in one week) is 0.75. But what is the probability that he will be in Town A after two steps (or even more steps)?

Notice that the 2-step transition probability from Town B to Town A is given by

$$\begin{aligned} & P(X_2 = A | X_0 = B) \\ = \ & P(X_1 = A | X_0 = B) P(X_2 = A | X_1 = A) \\ & + P(X_1 = B | X_0 = B) P(X_2 = A | X_1 = B) \\ & + P(X_1 = C | X_0 = B) P(X_2 = A | X_1 = C) \\ & = 0.75 \times 0 + 0 \times 0.75 + 0.25 \times 0.75 \\ & = 0.1875 \end{aligned}$$

Consider that

$$\begin{aligned} & [\mathbf{P} \times \mathbf{P}]_{BA} \\ = \ & \left\| \begin{array}{ccc} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{array} \right\| \cdot \left\| \begin{array}{ccc} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{array} \right\|_{BA} \end{aligned}$$

$$= \begin{Vmatrix} 0.75 & 0.125 & 0.125 \\ 0.1875 & 0.4375 & 0.375 \\ 0.1875 & 0.3750 & 0.4375 \end{Vmatrix}$$
$$= 0.1875.$$

The n-step transition probabilities can be computed using the Chapman-Kolmogorov equations.

Theorem For Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$, the n-step transition probability from state i to state j satisfies the **Chapman-Kolmogorov equation**

$$\begin{aligned} & p_{ij}^{(n+m)} \\ & = \sum_{k=0}^{\infty} P(X_n = k|X_0 = i)P(X_{m+n} = j|X_n = k) \\ & = \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)} \end{aligned}$$

or

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)}\mathbf{P}^{(n)}.$$
$$\mathbf{P}^{(n)} = \mathbf{P}^{n}.$$

Proof:

$$\begin{aligned} & p_{ij}^{(m+n)} \\ = \; & P(X_{m+n} = j|X_0 = i) \\ = \; & \sum_{k=0}^{\infty} P(X_{m+n} = j, X_n = k|X_0 = i) \\ = \; & \sum_{k=0}^{\infty} P(X_n = k|X_0 = i)P(X_{m+n} = j|X_n = k, X_0 = i) \\ = \; & \sum_{k=0}^{\infty} P(X_n = k|X_0 = i)P(X_{m+n} = j|X_n = k) \\ = \; & \sum_{k=0}^{\infty} P_{ik}^{(n)} p_{kj}^{(m)}. \end{aligned}$$

□

Example (salesman example continued) The 5-step transition probability matrix of the salesman example is given by

$$P^{(5)} = P^5$$

$$= \left\| \begin{matrix} 0.2930 & 0.3535 & 0.3535 \\ 0.5303 & 0.2344 & 0.235 \\ 0.5303 & 0.2354 & 0.2344 \end{matrix} \right\|$$

we have $P(X_5 = A|X_0 = B) = .5303$.

$$\begin{aligned} P^{(\infty)} &= \lim_{n\to\infty} P^n \\ &= \left\| \begin{matrix} 0.4286 & 0.2857 & 0.2857 \\ 0.4286 & 0.2857 & 0.2857 \\ 0.4286 & 0.2857 & 0.2857 \end{matrix} \right\| \end{aligned}$$

(what question you can ask about this limit?)

Some other facts

1. if $k_1 < k_2 < \cdots < k_m$, then

$$\begin{aligned} & P(X_{n+1} = j | X_{n-k_1} = i_{n-k_1}, \\ & \qquad X_{n-k_2} = i_{n-k_2}, \\ & \qquad \cdots, \\ & \qquad X_{n-k_m} = i_{n-k_m}) \\ & = P(X_{n+1} = j | X_{n-k_1} = i_{n-k_1}) \end{aligned}$$

2. a MC is determined by its one-step transition probability matrix and initial states.