

CHAPTER 3: (Discrete-time) Markov Chains

1 Definitions

Definition: Markov Chain Let $\{X_t : t \in T\}$ be a stochastic process with discrete-state space S and discrete-time space T satisfying

$$\begin{aligned} P(X_{n+1} = j | X_n = i, \\ X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ = P(X_{n+1} = j | X_n = i) \end{aligned}$$

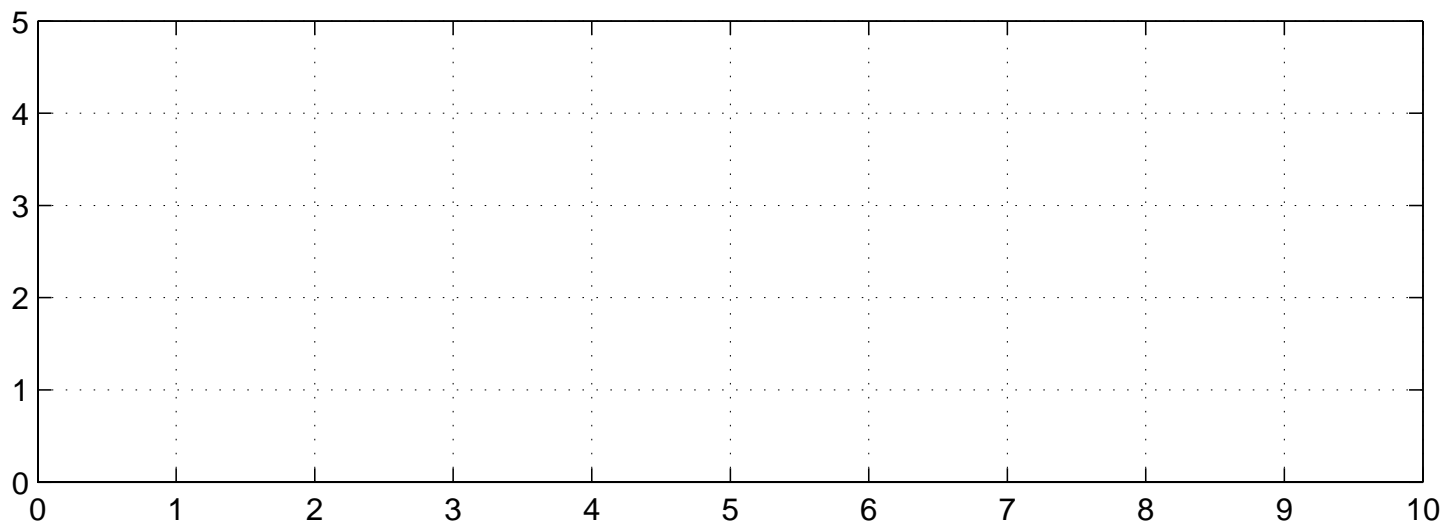
for any set of state $i_0, i_1, \dots, i_{n-1}, i, j$ in S and $n \geq 0$ is called a **Markov Chain (MC)**.

A process with the property stated in the above definition is said to have the **Markovian property**, i.e. the conditional distribution of any future state X_{n+1} depends on the present state and is independent of the past states.

Markov chain was named after A.A. Markov who developed the theoretical functions for the finite state Markov chains in the 1900s. An interesting example from the 19th century is called the Galton-Watson process which attempts to answer the question of when and with what probability would a given family name become extinct.

Example (Gambler's ruin (continued) with
N = 5)

$$P(X_{t+1} = j | X_t = i) = \begin{cases} 1/3, & j = i + 1 \text{ and } 5 > i > 0; \\ 2/3, & j = i - 1 \text{ and } 5 > i > 0; \\ 1, & \text{if } j = i = 0; \\ 1, & \text{if } j = i = 5; \\ 0, & \text{otherwise} \end{cases}$$



Let $P_{ij}^{n,n+1} = P(X_{n+1} = j | X_n = i)$, called **one-step transition probability**.

Definition: (**Stationary transition probability**) A Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ with state space S is said to have stationary transition probability if

$$\begin{aligned} & P(X_{n+1} = j | X_n = i) \\ &= P(X_n = j | X_{n-1} = i) \\ &= \dots \\ &= P(X_1 = j | X_0 = i) \end{aligned}$$

for each $i, j \in S$, i.e. probability of the one-step transition does not change as n increases.

NOTE: We consider Markov chains with stationary transition probability only.

A counter Example of stationary Markov Chain:

A promotion system T is the age; State space $\{junior, senior\}$, then

...

$$> P(X_{50} = s | X_{49} = j)$$

> ...

$$> P(X_{40} = s | X_{39} = j)$$

$$> P(X_{39} = s | X_{38} = j)$$

...

Transition probability matrix

$$\mathbf{P} = (p_{ij}) = \begin{array}{c|cccc} & 0 & 1 & 2 & \dots \\ \hline 0 & p_{00} & p_{01} & p_{02} & \dots \\ \hline 1 & p_{10} & p_{11} & p_{12} & \dots \\ \hline 2 & p_{20} & p_{21} & p_{22} & \dots \\ \hline \cdot & \dots & & & \dots \\ \hline \end{array}$$

Example (Gambler's ruin (continued) with $N = 5$)

$$P(X_{t+1} = j | X_t = i) = \begin{cases} 1/3, & j = i + 1 \text{ and } 5 > i > 0; \\ 2/3, & j = i - 1 \text{ and } 5 > i > 0; \\ 1, & \text{if } j = i = 0; \\ 1, & \text{if } j = i = 5; \\ 0, & \text{otherwise} \end{cases}$$

Properties of P

$$(1) \quad p_{ij} \geq 0, \quad \text{for } i, j \in S$$

$$(2) \quad \sum_{j=0}^{\infty} p_{ij} = 1, \quad \text{for } i \in S$$

—summation of each row in P is 1.

[Why:

]

State-diagram associated with transition probabilities —a very convenient way for us to visualize the process.

Example Draw a state diagram associated with the following transition probability matrix.

$$\mathbf{P} = \begin{vmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.2 & 0 & 0.1 & 0.7 \end{vmatrix}$$

Example (Gambler's ruin (continued))

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

States 0 and 5 are called **absorbing states** since once these states are reached, the process can never leave.

Example A salesman lives in Town A and is responsible for the sales consisting of Towns A, B and C. Each week he is required to visit a different town. When he is in his own town, it makes no difference which town he visits next so he slips a coin and if it is heads he goes to B and if tails he goes to C. However, after spending a week away from home, he has a slight preference for going home so when he is in either towns B or C, he has a slight preference for going home so when he is in either towns B or C, he flips two coins. If two heads occur, then he goes to the other town; otherwise he goes to A.

The successive towns that the salesman visits form a Markov chain with state space $\{A, B, C\}$ where random variable X_n equals A, B or C according to his location during week n . The transition probability matrix of this problem is given by

$$\mathbf{P} = \begin{vmatrix} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{vmatrix}$$

Given a Markov chain, what do we want to know next?

- After t steps, what's the probability that the MC is in state i ?
- As $t \rightarrow \infty$, what's the probability that the MC is in state i ?
- Given that we've taken t steps, what's the probability we've ever been in state i ?
- What's the expected number of steps before we reach state i for the first time?

2 Chapman-Kolmogorov Equations

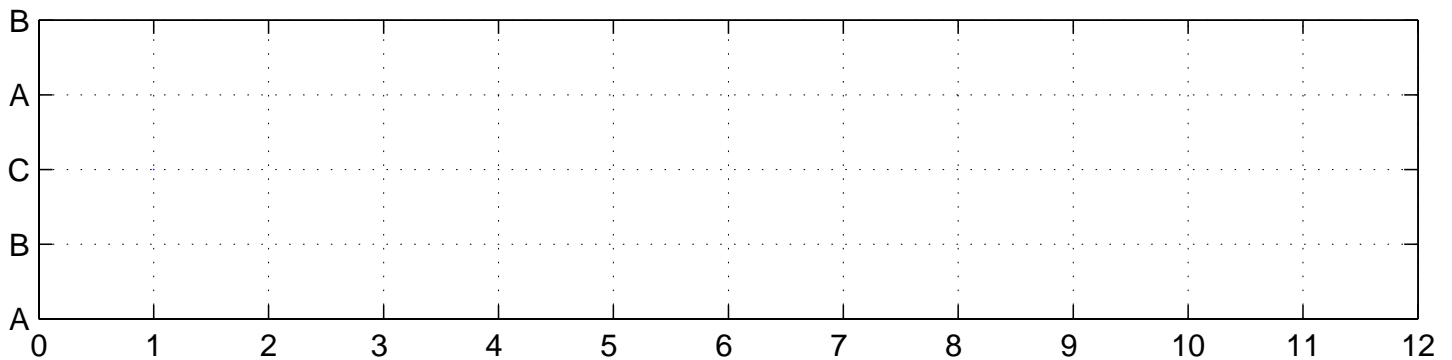
Denote the **n-step transition probability** of the Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ with state space S by

$$p_{ij}^{(n)} = P(X_{m+n} = j | X_m = i),$$
$$\forall m, n = 0, 1, 2, \dots, \quad \forall i, j \in S.$$

Similarly, we define the n-step transition probability matrix as

$$\mathbf{P}^{(n)} = (p_{ij}^{(n)}) = \begin{pmatrix} p_{00}^{(n)} & p_{01}^{(n)} & p_{02}^{(n)} & \cdots \\ p_{10}^{(n)} & p_{11}^{(n)} & p_{12}^{(n)} & \cdots \\ \vdots & & & \\ p_{i0}^{(n)} & p_{i1}^{(n)} & p_{i2}^{(n)} & \cdots \\ \vdots & & & \end{pmatrix}.$$

Consider the salesman example where the salesman starts from Town B. The transition probability matrix indicates that the probability of being in Town A after one step (in one week) is 0.75. But what is the probability that he will be in Town A after two steps (or even more steps)?



Notice that the 2-step transition probability from Town B to Town A is given by

$$P(X_2 = A | X_0 = B)$$

$$\begin{aligned}
&= P(X_1 = A|X_0 = B)P(X_2 = A|X_1 = A) \\
&\quad + P(X_1 = B|X_0 = B)P(X_2 = A|X_1 = B) \\
&\quad + P(X_1 = C|X_0 = B)P(X_2 = A|X_1 = C) \\
&= 0.75 \times 0 + 0 \times 0.75 + 0.25 \times 0.75 \\
&= 0.1875
\end{aligned}$$

Consider that

$$[\mathbf{P} \times \mathbf{P}]_{BA}$$

$$\begin{aligned}
&= \begin{vmatrix} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0.50 & 0.50 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{vmatrix}_{BA} \\
&= \begin{vmatrix} 0.75 & 0.125 & 0.125 \\ 0.1875 & 0.4375 & 0.375 \\ 0.1875 & 0.3750 & 0.4375 \end{vmatrix} \\
&= 0.1875.
\end{aligned}$$

The n-step transition probabilities can be computed using the Chapman-Kolmogorov equations.

Theorem: For Markov chain $\{X_n : n = 0, 1, 2, \dots\}$, the n-step transition probability from state i to state j satisfies the **Chapman-Kolmogorov equation**

$$\begin{aligned}
 & p_{ij}^{(n+m)} \\
 &= \sum_{k=0}^{\infty} P(X_n = k | X_0 = i) P(X_{m+n} = j | X_n = k) \\
 &= \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)}
 \end{aligned}$$

or

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)} \mathbf{P}^{(n)}.$$

$$\mathbf{P}^{(n)} = \mathbf{P}^n.$$

Proof:

$$\begin{aligned} & p_{ij}^{(m+n)} \\ &= P(X_{m+n} = j | X_0 = i) \\ &= \sum_{k=0}^{\infty} P(X_{m+n} = j, X_n = k | X_0 = i) \\ &= \sum_{k=0}^{\infty} P(X_n = k | X_0 = i) \\ & \quad \times \underline{P(X_{m+n} = j | X_n = k, X_0 = i)} \\ &= \sum_{k=0}^{\infty} P(X_n = k | X_0 = i) \underline{p(X_{m+n} = j | X_n = k)} \\ &= \sum_{k=0}^{\infty} P_{ik}^{(n)} p_{kj}^{(m)}. \end{aligned}$$

□

Example(salesman example continued) The 5-step transition probability matrix of the salesman example is given by

$$\begin{aligned}
 P^{(5)} &= P^5 \\
 &= \begin{vmatrix} 0.2930 & 0.3535 & 0.3535 \\ 0.5303 & 0.2344 & 0.235 \\ 0.5303 & 0.2354 & 0.2344 \end{vmatrix}
 \end{aligned}$$

we have $P(X_5 = A|X_0 = B) = .5303$.

$$\begin{aligned}
 P^{(\infty)} &= \lim_{n \rightarrow \infty} P^n \\
 &= \begin{vmatrix} 0.4286 & 0.2857 & 0.2857 \\ 0.4286 & 0.2857 & 0.2857 \\ 0.4286 & 0.2857 & 0.2857 \end{vmatrix}
 \end{aligned}$$

Some other facts

1. if $k_1 < k_2 < \dots < k_m$, then

$$\begin{aligned} &P(X_{n+1} = j | X_{n-k_1} = i_{n-k_1}, \\ &\quad X_{n-k_2} = i_{n-k_2}, \\ &\quad \dots, \\ &\quad X_{n-k_m} = i_{n-k_m}) \\ &= P(X_{n+1} = j | X_{n-k_1} = i_{n-k_1}) \end{aligned}$$

2. Show a MC is determined by its one-step transition probability matrix and initial states.
3. For a MC, given the present state, then the future states are independent of the past states