CHAPTER 6: Renewal Process (A generalization of Poisson Process)

1 Definition and notation

Suppose we have an infinite supply of light bulbs whose lifetimes $X_n, n \geq 1$ are iid (since the lightbulbs are of the same type, and each has nothing to do with the others). Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Let $N(t)$ be the number of lightbulbs that have failed by time t. $\{N(t), t \geq 0\}$ defined above is a renewal process

Definition Suppose that $X_1, \cdots, X_n...$ are iid random variable . Let $N(t)$ be the number of events that occurs before or at time t. Then $\{N(t), t \geq 0\}$ is called a **renewal process**.

Example A Bernoulli random walk is a renewal process in which the $X_n's$ are discrete: $P(X_n = 1) = 1 - P(X_n = 0)$.

Example Recall a poisson process counts the number of events in time (0, t] such that interarrive times $X_k, k = 1, 2, \cdots,$ are independent and follow a exponential distribution (with parameter λ).

Lemma 6.1. If the interarrive times $X_k, k = 1, 2, \cdots,$ are independent and follow a exponential distribution (with parameter λ), then $N(t)$ is a Poisson process of rate λ.

Basic properties of renewal processes

1. Waiting time
$$W_n = X_1 + \cdots + X_n, \quad n \geq 1$$
2. $\{N(t) \geq n\} \equiv \{W_n < t\}$.
3. $P(W_n < t) = P(X_1+X_2+...+X_n < t)$ [For Poisson process, W_n is a Gamma distribution.]

4. **Renewal function** $M(t) = E[N(t)]$

$$\begin{aligned} M(t) = E[N(t)] &= \sum_{k=1}^{\infty} P(N(t) \geq k) \\ = \sum_{k=1}^{\infty} P\{W_k \leq t\} &= \sum_{k=1}^{\infty} F_k(t). \end{aligned}$$

   Example If $\{X_n, n \geq 1\}$ are iid exponential random variables with parameter λ, then the renewal process becomes Poisson with $M(t) = \lambda t$.

5. Others

   (a) $\gamma_t = W_{N(t)+1} - t$

   (excess or residual lifetime)

   (b) $\delta_t = t - W_{N(t)}$

   (current life or age random variable)

   (c) $\beta_t = \gamma_t + \delta_t$ (total life)

2 Wald's Equation

Let $X_1, X_2, \cdots,$ denote a sequence of independent random variable s.

Definition AN integer-valued random variable N is said to be a **stopping time** for the sequence $X_1, X_2, \cdots,$ if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots,$.

Example let $X_n, n = 1, 2, \cdots,$ be independent and such that

$$P(X_n = -1) = P(X_n = 1) = 1/2$$

Then

$$N = \min\{n : X_1 + X_2 + ... + X_n = w\}$$

is a stoping time.

Theorem (Wald's Equation) Suppose N is a stoping time for iid sequence $X_1, X_2, \cdots,$ have the same distribution as X, and $EN < \infty$ then

$$E\sum_{n=1}^{N} X_n = E(N)E(X).$$

[Proof: Let

$$I_n = \begin{cases} 1, & \text{if } N \geq n \\ 0, & \text{if } N < n \end{cases}$$

We have

$$I_n = 1 \Longleftrightarrow \{ \text{ We have not stopped after successive on serving } X_1, \cdots, X_n \}$$

Thus I_n is determined by $X_1, \cdots, X_{n-1}$.

We have

$$\sum_{n=1}^{N} X_n = \sum_{k=1}^{\infty} X_n I_n$$

and

$$\begin{aligned} E\sum_{n=1}^{N} X_n &= \sum_{k=1}^{\infty} E(X_n I_n) \\ &= \sum_{k=1}^{\infty} E(X_n)E(I_n) \\ &= E(X)\sum_{k=1}^{\infty} E(I_n) \\ &= E(X)\sum_{k=1}^{\infty} P(N \geq n) \\ &= E(X)EN \end{aligned}$$

Corollary 3.3.3 if $\mu < \infty$, then

$$E(W_{N(t)+1}) = \mu[M(t) + 1].$$

[Proof:

$$W_{N(t)+1} = \sum_{i=1}^{N(t)+1} X_i$$

Now, $N(t)+1$ is a stoping time since

$$\{N(t)+1=n\} \Longleftrightarrow \left(\begin{array}{c} X_1+\cdots+X_{n-1} \leq t \\ \text{and } X_1+\cdots+X_n > t \end{array} \right)$$

By Wald's equation

$$\begin{aligned} E\{W_{N(t)+1}\} &= E\sum_{i=1}^{N(t)+1} X_i \\ &= E(N(t)+1)EX_i \\ &= \mu(M(t)+1). \end{aligned}$$

]

Example A miner is trapped in a room which contains three doors. Door 1 leads him to freedom after two-days travel; door 2 returns him to his room after four-days journey; and door 3 returns him to his room after six days travel. Suppose at all times he is equally likely to choose any of the 3 doors, and let S denote the time it takes the miner to become free. Define a sequence of iid r.v.s $X_1, X_2, ...$ and a stopping time T such that

$$S = \sum_{i=1}^{T} X_i$$

and find $E(S)$.

3 The poisson process as a special case of a renewal process

In poisson process, X_i has p.d.f.

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

The renewal function

$$M(t) = E[N(t)] = \lambda t$$

[Because

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2,$$

]

excess life $\gamma_t = W_{N(t)+1} - t$. Note that

$$\begin{aligned} & \{W_{N(t)+1} - t > x\} \\ & \Longleftrightarrow \text{No renewals in } (t, t+x] \\ & \Longleftrightarrow \{N(t+x) - N(t) = 0\} \end{aligned}$$

and

$$\begin{aligned} & P(N(t+x) - N(t) = 0) \\ & = P(N(x) = 0) = e^{-\lambda x}. \end{aligned}$$

Thus

$$P(\gamma_t > x) = e^{-\lambda x}.$$

Current life $\delta_t = t - W_{N(t)}$. For $x < t$,

$$\begin{aligned} & \{t - W_{N(t)+1} > x\} \\ & \Longleftrightarrow \text{No renewals in } (t-x, t] \\ & \Longleftrightarrow \{N(t) - N(t-x) = 0\} \end{aligned}$$

and by stationarity

$$P(N(t) - N(t-x) = 0) = P(N(x) = 0) = e^{-\lambda x}.$$

Thus, for $x < t$,

$$\begin{aligned} & P(\delta_t > x) = e^{-\lambda x}. \\ & P(\delta_t \le x) = 1 - e^{-\lambda x}. \end{aligned}$$

For $x > t$

$$P(\delta_t \le x) = 1.$$

Thus

$$P(\delta_t \le x) = \left\{ \begin{array}{lll} 1 - e^{-\lambda x} & for & 0 \le x < t \\ 1 & for & t \le x. \end{array} \right.$$

Mean Total life $\beta_t = \delta_t + \gamma_t$.

$$\begin{aligned} E(\beta_t) &= E(\gamma_t) + E(\delta_t) \\ &= \frac{1}{\lambda} + \int_0^t P(\delta_t > x) dx \\ &= \frac{1}{\lambda} + \int_0^t e^{-\lambda t} dx \\ &= \frac{1}{\lambda} + \frac{1}{\lambda}(1 - e^{-\lambda t}). \end{aligned}$$

An interesting observation is

$$E(\beta_t) > E(W_{k+1} - W_k) = \frac{1}{\lambda}.$$

Joint Distribution of γ_t and δ_t

$$\{\gamma_t > x, \delta_t > y\}$$

$$\Longleftrightarrow \{ \text{ Not renewals in } (t - y, t + x] \ \}$$

Thus

$$\begin{aligned} & P\{\gamma_t > x, \delta_t > y\} \\ &= \left\{ \begin{array}{l} e^{-\lambda(x+y)}, \text{ if } x > 0, 0 < y < t, \\ 0, \text{ if } y \ge t. \end{array} \right. \end{aligned}$$