Performance of recursive integration for pricing European-style Asian options

Tiong Wee Lim

Department of Statistics and Applied Probability, National University of Singapore

An Asian option is a security whose payoff depends on the price average of the underlying asset. It remains a challenge to be able to price an Asian option quickly and accurately, since no explicit pricing formula exists even for most European-style Asian options. In this paper, we demonstrate the method of recursive numerical integration for pricing European-style Asian options, for which early exercise is not allowed. The price average is written in such a way as to allow the evaluation of its density through a recursive sequence of one-dimensional integrals involving the univariate normal distribution. We highlight features of this implementation that are superior to competing methods. However, as recursive numerical integration can prove to be computationally intensive with a large number of fixings, we suggest two alternatives to circumvent this difficulty. The first uses a parametric approximation of the price average density; the other is based on the relationship between the number of fixings and the option price. As a by-product of the latter, we are able to deduce the price of a continuously monitored Asian option from a sequence of similar discretely monitored Asian options.

1. INTRODUCTION

An option is a security which gives its holder the right to receive a contingent payoff within a specified period of time. For an Asian option, this payoff depends on the price average of an underlying asset. Options utilizing the geometric average can be priced quite explicitly, since the geometric average of lognormal variates is itself lognormally distributed and Black-Scholes-type pricing formulas are easy to obtain. We will therefore focus on Asian options defined with the arithmetic average, considering both fixed strike and floating strike options, as well as both discretely fixed and continuously fixed options. We will demonstrate how the method of recursive numerical integration can be used to price European-style Asian options, for which early exercise is not allowed. This method has previously been applied to price barrier and lookback options by AitSahlia and Lai (1997, 1998).

Previous work on European-style Asian options can be placed into five broad categories. Kemna and Vorst (1990) demonstrated how *Monte Carlo simulation* can be used to price Asian options. Efforts involving *density approximation* include Carverhill and Clewlow (1990), who used Fourier transform to calculate the price average density, and Turnbull and Wakeman (1991) and Levy (1992), who relied on a lognormal approximation of the true distribution to obtain Black-Scholes-type pricing formulas. Analysis by *binomial tree* can be attributed to Barraquand and Pudet (1996), who improved on the proposed tree-based implementation of Hull and White (1993). Approaches using *finite-difference techniques* include Rogers and Shi (1995), Alziary, Décamps, and Koehl (1997), and Zvan, Forsyth, and Vetzal (1997), which differ mostly in the techniques used to solve a one-dimensional partial differential equation (PDE) in time and a Markovian state variable. Dewynne and Wilmott (1995), Andreasen (1998), and Tavella and Randall (2000) [Chapters 4 and 6] treated the case of discretely sampled Asian options. Using a combination of methods, Little and Pant (2000) used a numerical moments method to approximate the true distribution of the price average with the Johnson family of curves. Apart from numerical approaches to the problem, there are *quasi-analytic* methods. Geman and Yor (1993) derived an analytic expression for the Laplace transform in maturity for continuous Asian call options. Numerical inversion of this transform was considered in Geman and Eydeland (1995), Fu, Madan, and Wang (1998), and Craddock, Heath, and Platen (2000). Rogers and Shi (1995) applied a conditioning technique to obtain a numerical lower bound for option values.

The rest of the paper is organized as follows. In Section 2 we derive a sequence of recursive formulas that gives the density of the arithmetic average of lognormally distributed random variables and propose the method of recursive numerical integration for the evaluation of this density. We also develop a mixed density approximation to the true density of the arithmetic average. Both approaches allow us to compute certain expectations related to the pricing of Asian options with discrete fixings. These computations are carried out in Section 3. We price a variety of Asian options (both fixed strike and floating strike) and show that our price estimates are accurate against Monte Carlo simulation and compare favorably against a particular finite difference scheme. Numerical results also show that our mixed density approximation offers a substantial improvement over the more commonly used lognormal approximation. Next, we examine how the frequency of fixings affects the price of an option in Section 4. We suggest a simple approximation scheme that will improve the speed of our algorithm for real-time pricing of options. We conclude with comments in Section 5.

2. RECURSIVE NUMERICAL INTEGRATION

In the pricing of certain financial instruments, we have to evaluate expectations of the form $E[Y_m - \kappa]^+$ or $E[\kappa - Y_m]^+$, where Y_m is an arithmetic average given by

$$Y_m = (1 + e^{U_1} + \cdots + e^{U_m})/(m+1), \tag{1}$$

such that $U_0 = 0$, $U_j = X_1 + X_2 + \cdots + X_j$, and the X_i's are independent $N(\mu, \sigma^2)$ random variables. The last $i+1$ terms in the sum: $e^{U_{m-i}} + e^{U_{m-i+1}} + \cdots + e^{U_m}$, can be written in the form $e^{U_{m-i}} \times (i+1)Y_i$, where $Y_0 = 1$ and, for $i = 1, \ldots, m$,

$$Y_i = \frac{1 + e^{X_{m-i+1}}(1 + e^{X_{m-i+2}} + \cdots + e^{X_{m-i+2}+\cdots+X_m})}{i+1} \stackrel{\mathcal{L}}{=} \frac{1 + iY_{i-1}e^X}{i+1}. \tag{2}$$

Here, X is a $N(\mu, \sigma^2)$ random variable independent of Y_{i-1} and we write $A \stackrel{\mathcal{L}}{=} B$ to mean that A and B have the same distribution (law). Let f_i be the density of Y_i. It follows from (2) that f_m can be obtained using the following recursive integrals:

$$f_1(x) = \frac{1}{\sigma(x - h_1)}\, n\left(\frac{\log(x - h_1) - \log h_1 - \mu}{\sigma}\right), \quad x > h_1, \tag{3a}$$

$$f_i(x) = \int_{h_{i-1}}^{\infty} \psi_i(x, y) f_{i-1}(y)\, dy, \quad x > h_i, \quad i = 2, \ldots, m, \tag{3b}$$

where $h_i = 1/(i+1)$, $n(x) = e^{-x^2/2}/\sqrt{2\pi}$ (the standard normal density function), and

$$\psi_i(x, y) = \frac{1}{\sigma(x - h_i)}\, n\left(\frac{\log(x - h_i) - \log(yh_i/h_{i-1}) - \mu}{\sigma}\right). \tag{4}$$

The expectations can then be computed respectively as

$$E[Y_m - \kappa]^+ = \int_{\kappa}^{\infty} (x - \kappa) f_m(x)\, dx \quad \text{if } \kappa > h_m, \tag{5a}$$

$$E[Y_m - \kappa]^+ = \frac{h_m(e^{(m+1)(\mu+\sigma^2/2)} - 1)}{e^{\mu+\sigma^2/2} - 1} - \kappa \quad \text{if } \kappa \le h_m, \tag{5b}$$

and through a parity relation as

$$E[\kappa - Y_m]^+ = E[Y_m - \kappa]^+ + \kappa - \frac{h_m(e^{(m+1)(\mu+\sigma^2/2)} - 1)}{e^{\mu+\sigma^2/2} - 1}. \tag{6}$$

2.1 *Numerical Implementation*

The densities (3a)–(3b) have unbounded support. To implement a numerical scheme for computing these densities, we need to truncate the densities. For this purpose, we make two considerations. First, the value of each density outside its "effective range" should be negligible. Second, each truncated density should retain as much as possible of its coverage probability (i.e., the integral of a truncated density over its effective range should be as close to 1 as possible). When these requirements are imposed on the densities (3a)–(3b), the appropriate choices of "truncation points" (for fixed $0 < \alpha < 1$) are

$$x_i^L(\alpha) = h_i + h_i h_{i-1}^{-1} x_{i-1}^L(\alpha) \exp\{-\sigma A_i(\alpha) - (\sigma^2 - \mu)\},$$

$$x_i^R(\alpha) = h_i + h_i h_{i-1}^{-1} x_{i-1}^R(\alpha) \exp\{\sigma B_i(\alpha) - (\sigma^2 - \mu)\}, \quad i = 1, \ldots, m,$$

where $A_i(\alpha) = [B_i(\alpha)^2 + 2\log(x_{i-1}^R(\alpha)/x_{i-1}^L(\alpha))]^{1/2}$ and $B_i(\alpha) = \sigma + N^{-1}((1+\alpha)/2)$. Here, $N^{-1}(\cdot)$ is the standard normal inverse distribution function, given by the relation $\int_{-\infty}^{N^{-1}(\alpha)} n(x)\,dx = \alpha$. In these definitions, we set $x_0^L(\alpha) = x_0^R(\alpha) = 1$ for all α.

With the truncation points $x_i^L(\alpha)$ and $x_i^R(\alpha)$ so defined, we compute the densities (3a)–(3b) recursively as follows. For $\delta > 0$ (grid size) and $0 < \overline{\alpha} < 1$ (coverage probability of f_m), let $\alpha = \overline{\alpha}^{1/m}$, $k_i = \lfloor x_i^L(\alpha)/\delta \rfloor$, and $\ell_i = \lceil x_i^R(\alpha)/\delta \rceil$. When δ is sufficiently small, we can treat $f_i(y)$ as 0 for $y < k_i\delta$ and $y > \ell_i\delta$, and approximate $f_i(y)$ by $f_i(j\delta)$ for $(j - 1/2)\delta \le y < (j + 1/2)\delta$. Then we obtain $f_{i+1}(x)$ via the sum $\delta \sum_{j=k_i}^{\ell_i} \psi_{i+1}(x, y_j) f_i(y_j)$, where $y_j = j\delta$. Note that in this recursive algorithm, we need only compute $f_{i+1}(x)$ for $x = j\delta$, $j = k_{i+1}, \ldots, \ell_{i+1}$. Moreover, after the ith iteration we only have to store the $\ell_i - k_i + 1$ values of y_j and of $f_i(y_j)$ to be used in the next iteration. We estimate the expectation $E[Y_m - \kappa]^+$ with the sum $\delta \sum_{j=k_m}^{\ell_m} (j\delta - \kappa)^+ f_m(j\delta)$ and obtain $E[\kappa - Y_m]^+$ using (6).

2.2 *Density Approximation*

With sufficient computational time, especially for large m, recursive numerical integration (RNI) is able to produce very accurate estimates of the density f_m, hence of the expectations $E[Y_m - \kappa]^+$ and $E[\kappa - Y_m]^+$. In situations where time-critical analysis is necessary, it is feasible that RNI densities are computed ahead of time and stored for later retrieval in computations. An alternative approach would involve adopting a parametric approximation to the true density of the arithmetic average Y_m. There is numerical evidence to suggest that the density of an appropriately parametrized lognormal distribution offers a reasonably good approximation when σ is small. We propose a more general "mixed" density for this purpose.

The main inadequacy of the lognormal approximation for even moderately large σ is that it fails to account for *asymmetric tails* in the true distribution of $\log Y_m$. In order to incorporate this feature into our approximation, as well as to retain tractability of the approximate density, we assume that $\log Y_m$ is approximately distributed with a mixed density of the following form:

$$n^*(x) = \begin{cases} k(\nu_1, \nu_2, b_1, b_2) \exp\left[-\frac{1}{2}\left(\frac{a-x}{b_1}\right)^{\nu_1}\right], & x < a, \\ k(\nu_1, \nu_2, b_1, b_2) \exp\left[-\frac{1}{2}\left(\frac{x-a}{b_2}\right)^{\nu_2}\right], & x \geq a, \end{cases} \tag{7}$$

where $0 < \nu_2 < 2 < \nu_1$ and $k(\nu_1, \nu_2, b_1, b_2) = [b_1 \nu_1^{-1} 2^{1/\nu_1} \Gamma(1/\nu_1) + b_2 \nu_2^{-1} 2^{1/\nu_2} \Gamma(1/\nu_2)]^{-1}$. In (7), a is identified as the mode of the distribution, b_1 (resp. b_2) as the scale parameter of the left (resp. right) tail, and ν_1 (resp. ν_2) as the shape parameter of the left (resp. right) tail. In order to estimate the parameters $(a, \nu_1, \nu_2, b_1, b_2)$ and the normalizing constant $k(\nu_1, \nu_2, b_1, b_2)$, we transform the RNI density $f_m(\cdot)$ of Y_m to the "log-scale" density $n_m(\cdot)$ of $\log Y_m$, via the relation $n_m(x) = e^x f_m(e^x)$.

First, we obtain an estimate $\hat{a}$ of a using a quadratic fit to the log-scale density in the neighborhood of the mode. Denoting by (x_L, y_L), (x_0, y_0) and (x_R, y_R) the three points in this neighborhood, where $y_i = n_m(x_i)$ for $i = L, 0, R$, we have

$$\hat{a} = \frac{1}{2}\left[\frac{(x_0^2 - x_L^2)(y_0 - y_R) + (x_R^2 - x_0^2)(y_0 - y_L)}{(x_0 - x_L)(y_0 - y_R) + (x_R - x_0)(y_0 - y_L)}\right]. \tag{8}$$

From this we also obtain an estimate $\hat{k}$ for the normalizing constant $k(\nu_1, \nu_2, b_1, b_2)$:

$$\hat{k} = y_0 + (x_0 - \hat{a})^2 \left[\frac{y_0 - y_R}{(x_R - \hat{a})^2 - (x_0 - \hat{a})^2}\right]. \tag{9}$$

Next, it can be shown from (7) that

$$\log\left\{2 \log\left[\frac{k}{n^*(x)}\right]\right\} = \begin{cases} \nu_1 \log(a - x) - \nu_1 \log b_1, & x < a, \\ \nu_2 \log(x - a) - \nu_2 \log b_2, & x \geq a. \end{cases}$$

This suggests regressing $\log\{2 \log[\hat{k}/n_m(x)]\}$ against $\log(\hat{a} - x)$ for $x < \hat{a}$, and against $\log(x - \hat{a})$ for $x \geq \hat{a}$. From the respective least squares estimates of the gradients and vertical intercepts, denoted by $(\hat{m}_1^{LS}, \hat{c}_1^{LS})$ and $(\hat{m}_2^{LS}, \hat{c}_2^{LS})$, we obtain the estimates $(\hat{\nu}_1, \hat{\nu}_2, \hat{b}_1, \hat{b}_2)$ of (ν_1, ν_2, b_1, b_2):

$$\hat{\nu}_1 = \hat{m}_1^{LS}, \quad \hat{\nu}_2 = \hat{m}_2^{LS}, \quad \hat{b}_1 = \exp(-\hat{c}_1^{LS}/\hat{\nu}_1), \quad \hat{b}_2 = \exp(-\hat{c}_2^{LS}/\hat{\nu}_2). \tag{10}$$

Finally, the mixed density approximation (7), with the parameters duly estimated using (8)–(10), allows us to write down a closed-form expression for the expectation $E[Y_m - \kappa]^+$. We denote by $G_\alpha(\cdot)$ the standard gamma distribution function with shape parameter α, given by $G_\alpha(t) = \Gamma(\alpha)^{-1} \int_0^t x^{\alpha-1} e^{-x}\, dx$, and let $\overline{G}_\alpha(\cdot) = 1 - G_\alpha(\cdot)$.

PROPOSITION 1. *Assume that the density of* $\log Y_m$ *is given by* (7)*, with the parameters duly estimated using* (8)–(10)*. Then the expectation* $E_0 := E[Y_m - \kappa]^+$ *is given by*

$$E_0 = k\left\{e^a \sum_{i=0}^{\infty}\Big[(-1)^i u_{i1} G_{\eta_{i1}}(\xi_1) + u_{i2}\Big] - \kappa\Big[u_{01} G_{\eta_{01}}(\xi_1) + u_{02}\Big]\right\} \quad \textit{if} \;\; \kappa < e^a, \tag{11a}$$

$$E_0 = k\left\{e^a \sum_{i=0}^{\infty} u_{i2}\overline{G}_{\eta_{i2}}(\xi_2) - \kappa u_{02}\overline{G}_{\eta_{02}}(\xi_2)\right\} \quad \textit{if} \;\; \kappa \geq e^a, \tag{11b}$$

where k *is short for* $k(\nu_1, \nu_2, b_1, b_2)$*, and*

$$\xi_1 = \frac{1}{2}\left(\frac{a - \log\kappa}{b_1}\right)^{\nu_1}, \quad \xi_2 = \frac{1}{2}\left(\frac{\log\kappa - a}{b_2}\right)^{\nu_2},$$

$$\eta_{ij} = \frac{i+1}{\nu_j}, \quad u_{ij} = \frac{b_j^{i+1} 2^{\eta_{ij}} \Gamma(\eta_{ij})}{i!\,\nu_j}, \quad i = 0, 1, 2, \ldots, \quad j = 1, 2.$$

Proof of Proposition 1. Recall from (5a) that $E_0 = \int_{\log\kappa}^{\infty}(e^x - \kappa)n^*(x)\,dx$. Then, since $e^x = e^a \sum_{i=0}^{\infty}(x - a)^i/i!$, we have

$$E_0 = \left[e^a \sum_{i=0}^{\infty} \frac{1}{i!}\int_{\log\kappa}^{\infty}(x-a)^i n^*(x)\,dx - \kappa\int_{\log\kappa}^{\infty} n^*(x)\,dx\right]. \tag{12}$$

We now consider the integrals $\int_{\log\kappa}^{\infty}(x-a)^i n^*(x)\,dx$ ($i = 0, 1, 2, \ldots$). In the case $\kappa < e^a$, making use of the substitutions $u = [(a-x)/b_1]^{\nu_1}/2$ and $u = [(x-a)/b_2]^{\nu_2}/2$ respectively yields

$$\begin{aligned}\int_{\log\kappa}^{\infty}(x-a)^i n^*(x)\,dx &= k\int_{\log\kappa}^{a}(x-a)^i e^{-\frac{1}{2}(\frac{a-x}{b_1})^{\nu_1}}\,dx + k\int_{a}^{\infty}(x-a)^i e^{-\frac{1}{2}(\frac{x-a}{b_2})^{\nu_2}}\,dx \\ &= \frac{kb_1}{\nu_1}(-1)^i b_1^i 2^{\eta_{i1}}\int_0^{\xi_1} e^{-u}u^{\eta_{i1}-1}\,du + \frac{kb_2}{\nu_2} b_2^i 2^{\eta_{i2}}\int_0^{\infty} e^{-u}u^{\eta_{i2}-1}\,du \\ &= \frac{kb_1}{\nu_1}(-1)^i b_1^i 2^{\eta_{i1}}\Gamma(\eta_{i1})G_{\eta_{i1}}(\xi_1) + \frac{kb_2}{\nu_2} b_2^i 2^{\eta_{i2}}\Gamma(\eta_{i2}),\end{aligned} \tag{13a}$$

following the definition of $G_\alpha(\cdot)$ and the fact that $G_\alpha(t) \to 1$ as $t \to \infty$. In the other case $\kappa \geq e^a$,

$$\begin{aligned}\int_{\log\kappa}^{\infty}(x-a)^i n^*(x)\,dx &= k\int_{\log\kappa}^{\infty}(x-a)^i e^{-\frac{1}{2}(\frac{x-a}{b_2})^{\nu_2}}\,dx \\ &= \frac{kb_2}{\nu_2} b_2^i 2^{\eta_{i2}}\int_{\xi_2}^{\infty} e^{-u}u^{\eta_{i2}-1}\,du = \frac{kb_2}{\nu_2} b_2^i 2^{\eta_{i2}}\Gamma(\eta_{i2})\overline{G}_{\eta_{i2}}(\xi_2).\end{aligned} \tag{13b}$$

Equations (11a)–(11b) therefore follow by substituting (13a)–(13b) into (12).

3. ASIAN OPTIONS WITH DISCRETE FIXINGS

In the standard Black-Scholes environment, the asset price process is governed by the usual geometric Brownian motion, given by

$$S_t = S_0 \exp\{(r - v^2/2)t + vB_t\}, \quad S_0 > 0,$$

where r is the short-term interest rate, v is the instantaneous standard deviation of the asset return, and $\{B_t,\, t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure Q. In some cases, it proves more useful to change the numeraire from the riskless bond (as is the case for Black-Scholes) to the underlying asset. This is achieved through a change of measure from Q to $\tilde{Q}$ with $d\tilde{Q}/dQ = \xi_T$, where $\xi_t = e^{-rt}S_t/S_0$. As a consequence,

$$S_t = S_0 \exp\{(r + v^2/2)t + v\tilde{B}_t\}, \quad S_0 > 0,$$

where $\{\tilde{B}_t,\, t \geq 0\}$ is a standard Brownian motion under $\tilde{Q}$ given by $\tilde{B}_t = B_t - vt$.

For Asian options with discrete fixings, we assume that there are $m+1$ equally spaced monitoring dates between 0 and the expiration date T, and let $\Delta = T/m$.[1] In this setting, the initial values (i.e., at date 0) of European-style Asian options with fixed strike K are

$$C_0 = S_0 e^{-rT} E[Y_m - K/S_0]^+ \quad \text{and} \quad P_0 = S_0 e^{-rT} E[K/S_0 - Y_m]^+ \tag{14}$$

for the call and put respectively, where Y_m is the arithmetic average of $m+1$ discretely sampled prices (divided by S_0). Here, E denotes expectation with respect to the measure Q so the expectations in (14) are readily evaluated using (5a)–(5b) and (6) by setting $\kappa = K/S_0$, $\mu = (r - v^2/2)\Delta$, and $\sigma = v\sqrt{\Delta}$. Similarly, after a change of numeraire, the initial values of floating strike Asian options are

$$\tilde{C}_0 = S_0 \tilde{E}[1 - Y_m]^+ \quad \text{and} \quad \tilde{P}_0 = S_0 \tilde{E}[Y_m - 1]^+ \tag{15}$$

for the call and put respectively, where Y_m is the ratio of arithmetic price average to spot asset price; see Andreasen (1998), for example. Here, $\tilde{E}$ denotes expectation with respect to the measure $\tilde{Q}$. To make use of (5a)–(5b) and (6) to evaluate the expectations in (15), we set $\kappa = 1$, $\mu = -(r + v^2/2)\Delta$, and $\sigma = v\sqrt{\Delta}$. Thus, if we denote the fixed strike option prices by $C_0(S_0, K; r)$ for the call and $P_0(S_0, K; r)$ for the put, when the initial asset price is S_0, the strike price is K, and the riskless rate is r, then the preceding discussion shows that $\tilde{C}_0 = S_0 e^{-rT} P_0(1, 1; -r)$ and $\tilde{P}_0 = S_0 e^{-rT} C_0(1, 1; -r)$.

Our method is applied to a variety of Asian call options. We will compare the performance of RNI against Monte Carlo (MC) simulation, the forward shooting grid (FSG) algorithm of Barraquand and Pudet (1996), and the finite difference (FD) scheme of Tavella and Randall (2000) [Table 4.4]. As a simple validation of RNI, we note from (14) and (5b) that exact values can be obtained for all fixed Asian options with parameters

$$S_0 = 100,\ r = 0.09,\ T = 1,\ m = 52, \tag{16}$$

[1] The case of equally spaced fixings leads naturally to the case of continuous fixing as $m \to \infty$; see Section 4. The extension to nonuniform fixings is straightforward.

and $K < 100/53$. In particular, the exact price of zero-strike options is 95.63 for all volatilities. Correspondingly, evaluating the integral in (5a) using the RNI density and with $h_m = 1/53$ as the lower integration limit yields $C_0 = 95.63$ for zero-strike options with volatilities $v = 0.05, 0.10, 0.30, 0.50$. The maximum relative error in these four price estimates is 0.0024%, which shows that RNI can potentially produce remarkably accurate price estimates.

For the comparison with MC simulation, we price fixed strike Asian options with parameters (16), $K = 90, 95, 100, 105, 110$, and $v = 0.05, 0.10, 0.30, 0.50$. The RNI price estimates are reported in Table 1 under the columns headed RNI. Corresponding MC estimates, as reported in Levy and Turnbull (1992), are given in the columns headed MC. The two sets of estimates are remarkably close in value. To gauge the variability in price estimates obtained using MC simulation, we perform the following calculations with $K = 100$. For each volatility v we generate 1000 MC estimates, each based on a 10000 simulation series; see Kemna and Vorst (1990).[2] Summary statistics of the 1000 price estimates for each volatility are presented in Table 2. MC simulation produces relatively precise price estimates in the case of low volatility (e.g., $v = 0.10$). However, the precision of MC estimates deteriorates when v (or T) increases, as is evident from the substantially larger standard errors and wider ranges of estimates associated with higher volatility. We also report, for each volatility v, the percentage of MC estimates that are in error by an amount *more than* the corresponding RNI price estimate, taking the mean estimate (equivalent to an MC estimate based on a 10-million simulation series) to be the true value of the option. For example, when $v = 0.50$, we would obtain an MC estimate that is inferior to the RNI estimate 95.4% of the time! These measures suggest that RNI outperforms MC simulation when moderate to high volatility is present. The higher volatility requires more sampling repetitions to be run in simulation to obtain an estimate of a given precision, resulting in a loss of computational efficiency.

While Hull and White (1993) demonstrated the potential of tree methods for pricing Asian options, a computationally tractable algorithm was not available until Barraquand and Pudet (1996) implemented their FSG method. In Table 3 we compare RNI against the FSG algorithm for both fixed strike ($K = 95, 100, 105$) and floating strike Asian options, using parameters $S_0 = 100$, $r = 0.10$, $v = 0.10, 0.20, 0.40$, $m = 91$ (for $T = \frac{91}{365}, \frac{182}{365}$) or 121 (for $T = \frac{364}{365}$). MC estimates are also obtained and serve as the basis for comparison of the two methods. We find that the performance of the FSG algorithm is unsatisfactory for large v (or T). In such cases the FSG algorithm needs to be implemented with finer state quantization (at the expense of computational efficiency), a consequence of the Cox-Ross-Rubinstein binomial approximation it adopts. Accuracy was perhaps compromised in Barraquand and Pudet (1996) because their binomial walk

[2]To estimate the price of a discretely monitored option, we modify the variance reduction procedure in Kemna and Vorst (1990) slightly to use the theoretical value of a discretely fixed (instead of continuously fixed) Asian option with geometric averaging.

TABLE 1. Initial option values using MC simulation, RNI, and mixed density approximation (MDA).

K	$v=0.05$			$v=0.10$			$v=0.30$			$v=0.50$		
	MC	RNI	MDA	MC	RNI	MDA	MC	RNI	MDA	MC	RNI	MDA
90	–	13.379	13.373	–	13.386	13.381	14.96	14.963	14.962	18.14	18.144	18.113
95	8.81	8.810	8.808	8.91	8.910	8.912	–	11.631	11.635	–	15.395	15.379
100	4.31	4.308	4.309	4.91	4.909	4.911	8.81	8.801	8.811	12.98	12.980	12.979
105	0.95	0.954	0.955	2.06	2.061	2.064	–	6.490	6.506	–	10.882	10.896
110	–	0.051	0.051	–	0.624	0.627	4.68	4.670	4.692	9.10	9.079	9.105

NOTE: Discrete fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $T = 1$, and $m = 52$.
SOURCE: MC simulation results are due to Levy and Turnbull (1992).

TABLE 2. Summary statistics of 1000 MC simulation runs to estimate initial option values.

v	0.05	0.10	0.30	0.50
RNI estimate	4.3079	4.9088	8.8006	12.9795
Mean estimate (std err)	4.3080 (0.0005)	4.9090 (0.0012)	8.8015 (0.0087)	12.9808 (0.0254)
Range of MC estimates	4.3066–4.3096	4.9054–4.9127	8.7768–8.8289	12.9154–13.0636
Percent > RNI error	83.3%	87.3%	91.9%	95.4%

NOTE: Discrete fixed strike Asian call options with $S_0 = K = 100$, $r = 0.09$, $T = 1$, and $m = 52$.

employed steps that were too large.

For a comparison with the FD scheme of Tavella and Randall (2000),[3] we adopt the parameters $S_0 = 95, 100, 105$, $K = 100$, $r = 0.10$, $v = 0.40$, $T = 1$, $m = 10, 25, 50, 125, 250, 500, 1000$. Results presented in Table 4 show that price estimates obtained using RNI agree very well (to within a penny) with those obtained using the FD scheme under consideration.

RNI is designed to price options with discrete fixings. The key advantage of the method is that the accuracy and precision of price estimates do not deteriorate significantly as v (or T) increases, unlike MC simulation which loses precision and the FSG method which suffers from inaccuracy. *Once the terminal density f_m is obtained, option prices can be computed for any S_0 and K.* The computed densities can be stored for future real-time use, which makes for prompt valuation of Asian options. In addition, the density is useful for the evaluation of hedge parameters. For example, at the start of the averaging period, the delta

[3] In the notation of Section 2, Tavella and Randall (2000) defined the arithmetic average by $\tilde{Y}_m = (e^{U_1} + \cdots + e^{U_m})/m$, so $E[\tilde{Y}_m - \kappa]^+ = (1 + 1/m)E[Y_m - \tilde{\kappa}]^+$, where $\tilde{\kappa} = (1 + m\kappa)/(m + 1)$. Thus, the RNI price estimates are obtained in the usual way (i.e., via the density f_m of Y_m), but with an adjusted strike price, and then rescaled by a factor of $1 + 1/m$.

TABLE 3. Initial option values using MC simulation, the FSG method, and RNI.

$T,\ m$	K	$v=0.10$			$v=0.20$			$v=0.40$		
		MC	FSG	RNI	MC	FSG	RNI	MC	FSG	RNI
$\frac{91}{365}$, 91	95	6.116	6.132	6.115	6.472	6.500	6.469	8.093	8.151	8.083
	100	1.848	1.869	1.845	2.926	2.960	2.921	5.159	5.218	5.146
	105	0.147	0.151	0.146	0.944	0.966	0.939	3.054	3.106	3.043
	na	1.861	1.852	1.861	2.946	2.943	2.946	5.191	5.195	5.191
$\frac{182}{365}$, 91	95	7.215	7.248	7.213	7.884	7.793	7.879	10.333	10.425	10.318
	100	3.065	3.100	3.061	4.495	4.548	4.488	7.557	7.650	7.542
	105	0.711	0.727	0.706	2.197	2.241	2.191	5.359	5.444	5.344
	na	3.112	3.092	3.112	4.565	4.555	4.565	7.672	7.673	7.673
$\frac{364}{365}$, 121	95	9.278	9.313	9.275	10.291	10.336	10.276	13.705	13.825	13.690
	100	5.245	5.279	5.241	7.037	7.079	7.021	11.111	11.213	11.090
	105	2.288	2.313	2.282	4.506	4.539	4.488	8.901	8.989	8.879
	na	5.419	5.382	5.419	7.263	7.259	7.263	11.477	11.509	11.477

NOTE: Discrete Asian call options with $S_0 = 100$ and $r = 0.10$. $K =$ na indicates floating strike.
SOURCE: FSG results are due to Barraquand and Pudet (1996).

$\mathcal{D}_0$ and gamma $\mathcal{G}_0$ of a call option are given respectively by direct differentiation of (14) with (5a)–(5b):

$$\mathcal{D}_0 = e^{-rT}\int_{K/S_0}^{\infty} x f_m(x)\,dx \ \text{ and } \ \mathcal{G}_0 = e^{-rT}(K^2/S_0^3) f_m(K/S_0) \quad \text{if } \ K/S_0 > h_m,$$

$$\mathcal{D}_0 = \frac{h_m(e^{r\Delta} - e^{-rT})}{e^{r\Delta} - 1} \ \text{ and } \ \mathcal{G}_0 = 0 \quad \text{if } \ K/S_0 \le h_m.$$

For a fixed m the algorithm for RNI calls for the choice of two parameters: grid size δ and "final" coverage probability $\overline{\alpha}$. While a simple choice of $\overline{\alpha} = 0.9999$ is sufficient for most practical purposes, our choice of δ must provide enough grid points to capture accurately the shape of a recursive density $f_i(x)$ and the convoluting density given by (4) (particularly close to the modes) for the computation of the next density $f_{i+1}(x)$. In our implementation, we find that a good guide for this choice is to have about 50 grid points in the interval $(x_1^L(\alpha), x_1^R(\alpha))$. As a consequence of the trapezoidal integration scheme we have adopted, we find that our price estimates converge as $\delta \to 0$.

Finally, as an illustration of the mixed density approximation, we report in Table 5 the parameter estimates (8)–(10) for market parameters (16) and $v = 0.05,\ 0.10,\ 0.30,\ 0.50$. It is clear from these estimates that the density of $\log Y_m$ deviates more from symmetry as the volatility increases, which reiterates the inadequacy of the lognormal approximation. We also show that the mixed density approximation leads to accurate option prices against MC simulation. Using the setting of Table 1, we obtain price estimates of arithmetic average call options through the pricing formula (14) with (11a)–(11b), truncating the infinite

TABLE 4. Initial option values using the FD scheme and RNI.

m	T/m	$S_0 = 95$		$S_0 = 100$		$S_0 = 105$	
		FD	RNI	FD	RNI	FD	RNI
10	.1	9.2149	9.2223	12.0348	12.0394	15.2168	15.2241
25	.04	8.6974	8.7079	11.4803	11.4894	14.6415	14.6507
50	.02	8.5283	8.5362	11.2982	11.3062	14.4519	14.4596
125	.008	8.4304	8.4334	11.1929	11.1958	14.3424	14.3447
250	.004	8.3972	8.3990	11.1573	11.1587	14.3054	14.3063
500	.002	8.3804	8.3818	11.1392	11.1402	14.2866	14.2871
1000	.001	8.3719	8.3732	11.1300	11.1310	14.2771	14.2774
∞	.0	8.3640	8.3646	11.1215	11.1218	14.2681	14.2677

NOTE: Discrete fixed strike Asian call options with $K = 100$, $r = 0.10$, $v = 0.40$, and $T = 1$. Continuous option ($m = \infty$) values are computed by the RE-500 method (see Section 4).
SOURCE: FD results are due to Tavella and Randall (2000) [Table 4.4].

TABLE 5. Parameter estimates of the mixed density approximation.

v	Parameter estimates					
	$\hat{a}$	$\hat{\nu}_1$	$\hat{\nu}_2$	$\hat{b}_1$	$\hat{b}_2$	$\hat{k}$
0.05	0.04467	2.0129	1.9874	0.02894	0.02914	13.73
0.10	0.04261	2.0245	1.9738	0.05761	0.05835	6.87
0.30	0.02080	2.0677	1.9197	0.16816	0.17512	2.30
0.50	-0.02239	2.1084	1.8722	0.27050	0.29017	1.40

NOTE: $r = 0.09$, $T = 1$, and $m = 52$.

series after 100 terms. The results are summarized in Table 1 (on page 9) under the columns headed MDA. It is clear that the mixed density approximation leads to reasonably accurate option prices.

4. ASIAN OPTIONS WITH CONTINUOUS FIXINGS

Since all Asian options traded in the market are discretely fixed, it would be of interest to examine how the number of monitoring dates affects the price of an Asian option. Further, we will demonstrate how the price of continuously fixed Asian options can be approximated by the prices of corresponding options that are discretely fixed.

TABLE 6. Initial option values using RNI for various fixings m using RNI.

K	m						
	26	52	104	208	416	832	∞
90	14.9434	14.9629 (0.0195)	14.9728 (0.0099)	14.9779 (0.0051)	14.9803 (0.0024)	14.9815 (0.0012)	14.9827
95	11.6062	11.6306 (0.0244)	11.6427 (0.0121)	11.6487 (0.0060)	11.6518 (0.0031)	11.6533 (0.0015)	11.6548
100	8.7742	8.8006 (0.0264)	8.8143 (0.0137)	8.8210 (0.067)	8.8244 (0.0034)	8.8260 (0.0016)	8.8277
105	6.4631	6.4899 (0.0268)	6.5031 (0.0131)	6.5100 (0.0069)	6.5134 (0.0034)	6.5151 (0.0017)	6.5167
110	4.6459	4.6704 (0.0245)	4.6832 (0.0128)	4.6895 (0.0063)	4.6927 (0.0032)	4.6942 (0.0015)	4.6957

NOTE: Fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $v = 0.30$, and $T = 1$. Numbers in parentheses show increase in option price over the previous value. Continuous option ($m = \infty$) values are computed by the LS-6 method.

4.1 *Frequency of Fixings and Convergence of Option Values*

We first investigate how the price estimate varies with the number of monitoring dates m. As an illustration, we apply RNI to fixed strike Asian options with parameters $S_0 = 100$, $r = 0.09$, $v = 0.30$, $T = 1$, $K = 90, 95, 100, 105, 110$, and various numbers of monitoring dates: $m = 26, 52, 104, 208, 416, 832$. The results are recorded in Table 6. The estimates indicate strongly that there is a limiting option value when m gets infinitely large, since the increase in option price is halved whenever m is doubled. Indeed, we can view $A_T := T^{-1}\int_0^T S_u\,du$ as the limit, as $m \to \infty$, of its discrete counterpart $A_m = S_0 Y_m$, where Y_m is of the form (1). The rate at which discrete Asian option prices converge to the continuous prices would be of interest. The following convergence result for fixed strike discrete geometric average options is useful.

PROPOSITION 2. *The value c_0 of a discrete fixed strike geometric average call option converges to the value c_0^* of a corresponding continuous option at the rate of $O(m^{-1})$. Specifically, for m large,*

$$4m(c_0^* - c_0) \sim S_0 e^{-(r+v^2/6)T/2}[\sigma^{*2} N(d^* + \sigma^*) - (d^* - \sigma^*)n(d^* + \sigma^*)] + Ke^{-rT} d^* n(d^*),$$

where $\sigma^* = v\sqrt{T/3}$, $d^* = \sigma^{*-1}[\log(S_0/K) + (r - v^2/2)T/2]$, and $N(x) = \int_{-\infty}^x n(t)\,dt$.

Proof of Proposition 2. Since the discrete geometric average is lognormally distributed with mean $\log S_0 + (r - v^2/2)T/2$ and standard deviation $\sigma_m = v\sqrt{T(2m+1)/6(m+1)}$, the Black-Scholes formula gives

$$c_0 = S_0 e^{\sigma_m^2/2 - (r+v^2/2)T/2} N(d_m + \sigma_m) - KN(d_m),$$

where $d_m = \sigma_m^{-1}[\log(S_0/K) + (r - v^2/2)T/2]$. A similar argument shows that c_0^* is given by (essentially) the same expression, except with σ_m and d_m replaced by σ^* and d^*, respectively. The result follows when we use the following Taylor series approximations to the difference $c_0^* - c_0$ when m is large:

$$e^{(\sigma_m^2 - \sigma^{*2})/2} \doteq e^{-\sigma^{*2}/4(m+1)} \doteq 1 - \sigma^{*2}/4m,$$

$$N(d_m) \doteq N(d^* + d^*/4(m+1)) \doteq N(d^*) + d^* n(d^*)/4m,$$

$$N(d_m + \sigma_m) \doteq N(d^* + \sigma^* + (d^* - \sigma^*)/4(m+1)) \doteq N(d^* + \sigma^*) + (d^* - \sigma^*) n(d^* + \sigma^*)/4m.$$

Proposition 2 suggests approximating the option price for large m, holding all other parameters constant, by a function of the form $C_0^* - \beta m^{-1}$, where $C_0^* > 0$ and β are parameters to be determined. In particular, C_0^* is the price of the Asian option when the average is computed continuously since $m^{-1} \to 0$ as $m \to \infty$. For the Asian options considered in Table 6, their continuous values are reported in the final column using the scheme LS-6 to be described shortly. We note that in these examples, the price of an option with weekly fixings differs by no more than 0.5% from the price of a corresponding option with continuous fixings.

One simple scheme for estimating the constants C_0^* and β consists of evaluating the option prices using RNI based on m and $2m$ monitoring dates (denote these prices by C_1 and C_2 respectively), and then solving the two equations that result: $C_k = C_0^* - \beta(km)^{-1}$ ($k = 1, 2$), to give $C_0^* = 2C_2 - C_1$ and $\beta = 2m(C_2 - C_1)$. This method is commonly known as Richardson's extrapolation. Implementing this scheme with $m = 104$ and $m = 416$ for the Asian options considered in Table 6 yields estimates respectively under the columns headed RE-104 and RE-416 in Table 7. Another approach is to regress the RNI estimates against m^{-1} and obtain C_0^* and $-\beta$ respectively as the least squares estimates of the intercept and gradient for the regression line. Results using the regression approach are reported under the columns headed LS-3 (based on RNI estimates corresponding to $m = 26, 52, 104$) and LS-6 (based on all six RNI estimates in Table 6). It is clear from Table 7 that using RNI values with $m \leq 104$ produces sufficiently accurate estimates of C_0^* and β, leading to substantial savings in computational time.

4.2 *Comparison with Finite-Difference and Quasi-Analytic Approaches*

By exploiting a scaling property of Brownian motion, Rogers and Shi (1995) showed that the pricing problem can be reduced to solving a one-dimensional parabolic PDE, for which an accurate algorithm employing a high-order nonlinear flux limiter for the convection term was implemented by Zvan, Forsyth, and Vetzal (1997). Moreover, a convenient lower bound of the option price was given. A comparison of our price estimates with the results of Rogers and Shi (1995) and Zvan, Forsyth, and Vetzal (1997) is presented in Table 8 with $S_0 = 100$, $T = 1$ and various choices of (r, v, K). Our estimates of the continuous option values are based on the LS-3 scheme (using RNI estimates with $m = 26, 52, 104$). All three methods yield price estimates that are practically identical in *all* cases we consider. We conclude that our simple scheme

TABLE 7. Estimated initial option values C_0^* and coefficients β using Richardson's extrapolation (RE-104, RE-416) and least squares regression (LS-3, LS-6) on a set of RNI estimates.

K	RE-104		RE-416		LS-3		LS-6	
	C_0^*	β	C_0^*	β	C_0^*	β	C_0^*	β
90	14.9829	1.04465	14.9828	1.00608	14.9826	1.01900	14.9827	1.02495
95	11.6547	1.24816	11.6547	1.21612	11.6549	1.26400	11.6548	1.26212
100	8.8276	1.37946	8.8277	1.35312	8.8275	1.38659	8.8277	1.39228
105	6.5169	1.43085	6.5168	1.40900	6.5165	1.38755	6.5167	1.39397
110	4.6957	1.29682	4.6958	1.27839	4.6955	1.29179	4.6957	1.29969

NOTE: Continuous fixed strike Asian call options with $S_0 = 100$, $r = 0.09$, $v = 0.30$, and $T = 1$.

for evaluating continuous option values performs rather well for all parameter values.[4]

The implication of our observation goes beyond a simple procedure for estimating the price of continuously monitored Asian options. The method of RNI can become inefficient when the number of fixings m gets increasingly large. However, the approximate formula $C_0^* - \beta m^{-1}$, with the constants C_0^* and β estimated using either Richardson's extrapolation or least squares regression, allows us to estimate the price of an option with *any* number of monitoring dates, provided we have the RNI estimates needed for either scheme to be implemented. As we have demonstrated, C_0^* and β can be estimated using relatively small numbers of monitoring dates, making the amount of computational time acceptable.

5. CONCLUSION

We have presented a *unified approach* for the pricing of both discrete and continuous Asian options. The method of recursive numerical integration is applicable to the pricing of both fixed strike and floating strike discrete Asian options. Straightforward extensions of the basic method allow us to price the options other than at their inception, such as prior to the averaging period, or into the averaging period. By virtue of the put-call parity relation (6), all put options are also covered. The key advantage of this method, other than computational efficiency, is that the accuracy of RNI price estimates is not sensitive to changes in the volatility and/or time to maturity. Moreover, the RNI densities are themselves useful for the determination of a mixed density approximation for the true density of the discrete price average, thereby leading to *new approximations* for Asian option values.

More importantly, we demonstrate that there exists a *quantitative* relationship between the price of

[4]We add that the PDE solutions of Zvan, Forsyth, and Vetzal (1997) dominate the estimates of Alziary, Décamps, and Koehl (1997) and that our estimates compare favorably with the option values of Little and Pant (2000).

TABLE 8. Initial option values using lower bound (LB), PDE solution, and RNI.

v	K	$r = 0.05$		$r = 0.09$		$r = 0.15$		
		LB	RNI	LB	RNI	PDE	LB	RNI
0.05	95	7.178	7.177	8.809	8.809	11.094	11.094	11.094
	100	2.716	2.716	4.308	4.308	6.793	6.794	6.794
	105	0.337	0.337	0.958	0.958	2.748	2.744	2.744
0.10	90	11.951	11.951	13.385	13.385	15.399	15.399	15.398
	100	3.641	3.641	4.915	4.915	7.030	7.028	7.027
	110	0.331	0.331	0.630	0.630	1.410	1.413	1.413
0.20	90	12.595	12.596	13.831	13.831	15.643	15.641	15.641
	100	5.762	5.762	6.777	6.776	8.409	8.408	8.408
	110	1.989	1.989	2.545	2.545	3.554	3.554	3.554
0.30	90	13.952	13.953	14.983	14.983	16.514	16.512	16.512
	100	7.944	7.946	8.827	8.829	10.210	10.208	10.210
	110	4.070	4.071	4.695	4.696	5.729	5.728	5.730

NOTE: Continuous fixed strike Asian call options with $S_0 = 100$ and $T = 1$.
SOURCE: LB from Rogers and Shi (1995). PDE from Zvan, Forsyth, and Vetzal (1997).

an Asian option and the number of fixings. This relationship can be capitalized to yield option prices in the case of continuous fixings. Our approach calls for the use of RNI price estimates in the evaluation of the continuous option prices, either via Richardson's extrapolation or by regression. Along with the continuous option values, we also solve for the relationship between option values and fixing frequency, thereby enabling us to price Asian options with *any number of fixings*.

REFERENCES

AitSahlia, F. and Lai, T. L. (1997). Valuation of discrete barrier and hindsight options. *Journal of Financial Engineering*, **6**, 169–177.

AitSahlia, F. and Lai, T. L. (1998). Random walk duality and the valuation of discrete lookback options. *Applied Mathematical Finance*, **5**, 227–240.

Alziary, B., Décamps, J.-P., and Koehl, P.-F. (1997). A P.D.E. approach to Asian options: Analytical and numerical evidence. *Journal of Banking and Finance*, **21**, 613–640.

Andreasen, J. (1998). The pricing of discretely sampled Asian and lookback options: A change of numeraire approach. *Journal of Computational Finance*, **2**, 5–30.

Barraquand, J. and Pudet, T. (1996). Pricing of American path-dependent contingent claims. *Mathematical Finance*, **6**, 17–51.

Carverhill, A. P. and Clewlow, L. J. (1990). Flexible convolution. *Risk*, **3**, 25–29.

Craddock, M., Heath, D., and Platen, E. (2000). Numerical inversion of Laplace transforms: A survey of techniques with applications to derivative pricing. *Journal of Computational Finance*, **4**, 57–81.

Dewynne, J. N. and Wilmott, P. (1995). A note on average rate options with discrete sampling. *SIAM Journal on Applied Mathematics*, **55**, 267–276.

Fu, M. C., Madan, D. B., and Wang, T. (1998). Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance*, **2**, 49–74.

Geman, H. and Eydeland, A. (1995). Domino effect: Inverting the Laplace transform. *Risk*, **8**, 65–68.

Geman, H. and Yor, M. (1993). Bessel processes, Asian options, and perpetuities. *Mathematical Finance*, **3**, 349–375.

Hull, J. C. and White, A. (1993). Efficient procedures for valuing European and American path-dependent options. *Journal of Derivatives*, **1**, 21–31.

Kemna, A. G. Z. and Vorst, A. C. F. (1990). A pricing method for options based on average asset values. *Journal of Banking and Finance*, **14**, 113–129.

Levy, E. (1992). Pricing European average rate currency options. *Journal of International Money and Finance*, **14**, 474–491.

Levy, E. and Turnbull, S. M. (1992). Average intelligence. *Risk*, **5**, 53–59.

Little, T. and Pant, V. (2000). A PDE method for computing moments. *Journal of Computational Finance*, **4**, 5–20.

Rogers, L. C. G. and Shi, Z. (1995). The value of an Asian option. *Journal of Applied Probability*, **32**, 1077–1088.

Tavella, D. and Randall, C. (2000). *Pricing Financial Instruments: The Finite Difference Method*. Wiley Series in Financial Engineering, New York: John Wiley & Sons.

Turnbull, S. M. and Wakeman, L. M. (1991). A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, **26**, 377–389.

Zvan, R., Forsyth, P. A., and Vetzal, K. R. (1997). Robust numerical methods for PDE models of Asian options. *Journal of Computational Finance*, **1**, 39–78.