ELSEVIER

Statistics & Probability Letters 46 (2000) 105–112

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

Marcinkiewicz strong laws for linear statistics

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Received October 1998; received in revised form March 1999

Abstract

Strong laws are established for linear statistics that are weighted sums of a random sample. We show extensions of the Marcinkiewicz–Zygmund strong law under certain moment conditions on both the weights and the distribution. These complement the results of Cuzick (1995, J. Theoret. Probab. 8, 625–641) and Bai et al. (1997, Statist. Sinica, 923–928).

MSC: 60F15; 62G05

Keywords: Hardy–Littlewood strong law; Marcinkiewicz–Zygmund strong law; Weighted sums of i.i.d. random variables

1. Introduction

Many useful linear statistics based on a random sample are weighted sums of i.i.d. random variables. Examples include least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. In this respect, studies of strong laws for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics.

Let $X_i,\ i \geqslant 1$, be a sequence of independent observations from a population distribution. A common expression for these linear statistics is

$$T_n = \sum_{i=1}^{n} a_{ni} X_i, \tag{1.1}$$

where the weights a_{ni} are either real constants or random variables independent of X_i. Using an observation of the Bernstein inequality by Cheng (1995), Bai et al. (1997) recently established an extension of the Hardy–Littlewood strong law for linear statistics T_n. This complements a result of Cuzick (1995, Theorem 2.2), who also gave an interesting result of (1.2) below for the case $0 < p \leqslant 1$ with $b_{n,p} = n^{1/p}$. Typical forms of

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PII: S0167-7152(99)00093-0

such results are expressed by either

$$T_n/b_{n,p} \to 0 \quad \text{a.s.} \tag{1.2}$$

or

$$\limsup_{n\to\infty} |T_n|/b_{n,p} \leqslant t^* < \infty \quad \text{a.s.} \tag{1.3}$$

for $1\leqslant p\leqslant 2$ with some suitable normalizing constants $b_{n,p}$. Specifically, assume the double array weights $\{a_{ni},\ 1\leqslant i\leqslant n,\ n\geqslant 1\}$ satisfy that $A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty$, $A^\alpha_{\alpha,n} = n^{-1}\sum_{i=1}^n |a_{ni}|^\alpha$, and $E|X|^\beta < \infty$, $EX = 0$, $1<\alpha,\ \beta\leqslant\infty$. Setting $1/p = 1/\alpha + 1/\beta$ and $b_{n,p} = n^{1/p}(\log n)^{1-1/p}$, Bai et al. (1997) established a result (1.2) when $1<p<2$, and (1.3) with $t^* = \sqrt{2}A_2(E^{1/2}|X|^2)$ when $p=2$.

In case $1<p<2$, the above result (1.2) of Bai et al. (1997) is non-optimal in light of the classical Marcinkiewicz–Zygmund (MZ) strong law and Cuzick's Theorem 2.2 which assumes uniformly bounded weights a_{ni} ($\alpha=\infty$) and a condition on the tail distribution of X. Remark that by setting $a_{nn} = n^{1/\alpha}$ and $a_{ni}=0$ for $1\leqslant i<n$, the condition $E|X|^\beta<\infty$ is seen to be necessary, if for any weights with $A_\alpha<\infty$ the conclusion $T_n/n^{1/p}\to 0$ a.s. holds.

In view of an enlightening example (Cuzick, 1995, Example 2.1) that the extended MZ strong law fails to hold when $\alpha=\infty$ and $1<p<2$, our goal in this study is to establish the sufficiency of the moment conditions $A_\alpha<\infty$ and $E|X|^\beta<\infty$, $1<\alpha,\ \beta<\infty$. Complementing the situations with finite-moments conditions, we also wish to consider a standard case when the moment generating function of the variable X exists. By the study of Bai et al. (1997), we find that it suffices to consider this case with finite A_α, $\alpha\in(0,2)$. The results will be presented in Section 2, and the proofs will be detailed in Section 3.

2. The MZ strong laws

Let $T_n = \sum_{i=1}^n a_{ni}X_i$ be a weighted sum of (1.1), where X_i, $i\geqslant 1$, are i.i.d. random variables with mean 0. In the sequel, set $1/p = 1/\alpha + 1/\beta$, for $1<\alpha,\ \beta<\infty$ and $1<p<2$. Assume the moment condition defined after (1.3), specifically,

$$A_\alpha = \limsup A_{\alpha,n} < \infty. \tag{2.1}$$

Theorem 2.1. *Let $T_n = \sum_{i=1}^n a_{ni}X_i$, $n\geqslant 1$, be a weighted sum of* (1.1). *Suppose* (2.1) *holds and for some $1<p<2$ and $1<\alpha,\ \beta<\infty$, $E|X|^\beta<\infty$, and $EX=0$. Then*

$$T_n/n^{1/p} \to 0 \quad a.s. \tag{2.2}$$

Conversely, if (2.2) *is true for any coefficient arrays satisfying* (2.1), *then $E|X|^\beta<\infty$, and $EX=0$.*

Remark 1. In connection to our previous study, Theorem 2.1 complements the results (1.2) and (1.3), when condition (2.1), $E|X|^\beta<\infty$, and the identity $1/\alpha+1/\beta=1/p$, $1<p\leqslant 2$, hold. Recall that when $p=2$, hence $\alpha\wedge\beta\geqslant 2$, (1.3) gave a sharp upper law with rate $b_{n,2}$ unless an upper iterated logarithm law holds under further conditions. For $1<p<2$, we now establish that either the MZ strong law (2.2) holds when $\max\{\alpha,\beta\}<\infty$, or (1.2) holds when either α or β equals to ∞.

To further this study, let us consider another standard case of interest where the random variable X, or for ease of exposition, $|X|^\gamma$, $0<\gamma$, has a finite moment generating function. This corresponds to the case with an arbitrarily large β in Theorem 2.1, and thus, the next theorem complements the results of Cuzick (1995). Specifically, we assume that for some $h,\ \gamma>0$

$$E\exp(h|X|^\gamma) < \infty. \tag{2.3}$$

Theorem 2.2. *Let* $T_n = \sum_{i=1}^n a_{ni}X_i$, $n \geqslant 1$, *be a weighted sum of* (1.1). *Suppose* (2.1) *holds for* $\alpha \in (0,2)$ *and* (2.3) *holds. Then for* $0 < \alpha \leqslant 1$ *and* $b_n = n^{1/\alpha} \log^{1/\gamma} n$

$$\limsup |T_n|/b_n \leqslant h^{-1/\gamma} A_\alpha \quad a.s., \tag{2.4}$$

moreover, for $1 < \alpha < 2$, $b_n = n^{1/\alpha}(\log n)^{1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)}$, *and* $EX = 0$,

$$\limsup |T_n|/b_n = 0 \quad a.s. \tag{2.5}$$

Conversely, if (2.4) *holds when* $0 < \alpha \leqslant 1$ *for all weights sequences satisfying* (2.1), *then necessarily* (2.3) *holds,* $E(h'|X|^\gamma) < \infty$ *for any* h', $0 < h' < h$; *if* (2.5) *holds when* $1 < \alpha < 2$, *then* $E(h|X|^\eta) < \infty$ *for any* $h > 0$, *where* $1/\eta = 1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)$.

Remark 2. It seems less satisfactory that the necessary condition for (2.4) is not exactly (2.3) when $0 < \alpha \leqslant 1$. This is due to the uncertainty of the lim sup and that of the sequence $A_{\alpha,n}$. If we present (2.4) as "with probability one, there are at most finitely many n such that $|T_n|/b_n \geqslant h^{-1/\gamma}A_{\alpha,n}$", then the necessary and sufficient condition is exactly (2.3). Likewise, for (2.5) when $1 < \alpha < 2$, we have not found a satisfactory necessary condition; but by analogous argument, the converse (2.3) holds only when $\eta < \gamma$ as specified above. However, we believe the choice of b_n can hardly be improved in view of Cuzick (1995), Lemma 2.1), since the limit $b_n = n^{1/\alpha} \log^{1-1/\alpha} n$ obtains as $\gamma \to \infty$.

3. Proofs

3.1. *Proof of the sufficiency part of Theorem 2.1*

Let $X_i^c = X_i I[|X_i|^\beta > n]$, $X_i' = X_i I[(n^{1/\beta}/(\log n)^{3(\alpha-1)}) < |X_i| \leqslant n^{1/\beta}]$, and $\bar{X}_i = X_i I[|X_i| \leqslant n^{1/\beta}/(\log n)^{3(\alpha-1)}]$, for all $1 \leqslant i \leqslant n$, where the dependence on the sample size n is suppressed. For $1 < \alpha < \infty$, define $a'_{ni} = a_{ni} I[|a_{ni}| > n^{1/\alpha}/\log^2 n]$ and $\bar{a}_{ni} = a_{ni} - a'_{ni}$, for all $1 \leqslant i \leqslant n$. Set $T_n^c = \sum_{i=1}^n a_{ni}X_i^c$, $T_n' = \sum_{i=1}^n a'_{ni}X_i'$, $T_n^* = \sum_{i=1}^n a'_{ni}\bar{X}_i$, and $\bar{T}_n = \sum_{i=1}^n \bar{a}_{ni}(X_i' + \bar{X}_i) = \sum_{i=1}^n \bar{a}_{ni}(X_i - X_i^c)$. By definition

$$T_n = T_n^c + T_n' + T_n^* + \bar{T}_n. \tag{3.1}$$

We show first that $T_n^c/n^{1/p} \to 0$ a.s. Since $1/p = 1/\alpha + 1/\beta$, $\beta = \alpha/(\alpha-1)\{1 + \beta(1-1/p)\}$, it holds that

$$|X_i^c| \leqslant |X_i^c|^{\beta(\alpha-1)/\alpha} n^{-(1-1/p)}.$$

By the Hölder inequality and the moment condition $E|X|^\beta < \infty$,

$$n^{-1/p}|T_n^c| \leqslant n^{-1} \sum_{i=1}^n |a_{ni}||X_i^c|^{\beta(\alpha-1)/\alpha} \leqslant A_{\alpha,n} \left(\frac{1}{n} \sum_{i=1}^n |X_i^c|^\beta \right)^{(\alpha-1)/\alpha} \to 0 \quad \text{a.s.} \tag{3.2}$$

Next, for T_n', by the definitions of $A_{\alpha,n}$ and X_i', we have

$$\begin{aligned} n^{-1/p}|T_n'| &\leqslant n^{-1/p} \left\{ \max_{1 \leqslant i \leqslant n} |a'_{ni}X_i'| + \left(\sum_{i=1}^n |a'_{ni}X_i'| \right) I[\#\{i : a'_{ni}X_i' \neq 0\} \geqslant 2] \right\} \\ &\leqslant n^{-1/\beta} \max_{1 \leqslant i \leqslant n} |X_i| + n^{1-1/\alpha} I[\#\{i: a'_{ni}X_i' \neq 0\} \geqslant 2]. \end{aligned} \tag{3.3}$$

Since $E|X|^\beta < \infty$, the first term of the last sum converges to zero a.s. For the last term on the r.h.s., there are at most $\mathrm{O}(\log^{2\alpha} n)$ many nonzero a'_{ni} by the definitions of $A_{\alpha,n}$ and a'_{ni}, and so, by the definition of X'_i,

$$P([\#\{i : a'_{ni}X'_i \neq 0\} \geqslant 2]) \leqslant P\left(\bigcup_{i\neq j}\{a'_{ni}X'_i \neq 0, a'_{nj}X'_j \neq 0\}\right)$$

$$\leqslant (\mathrm{O}(\log n))^{4\alpha} P^2(|X_i| > n^{1/\beta}(\log n)^{-3(\alpha-1)}) = \mathrm{O}((\log n)^{4\alpha+6\beta(\alpha-1)})n^{-2}.$$

By this estimate and Markov's inequality, the probability that the last term of (3.3) is positive is a general term of a convergent series in $n \geqslant 1$. Therefore, the l.h.s. of (3.3) converges to zero a.s. Furthermore, associated with (3.2) and (3.3), we find that

$$n^{-1/p}|E(T_n^c + T'_n)| \leqslant \frac{n^{-1/p}E|X|^\beta \left(\sum_{i=1}^n |a_{ni}|\right)}{(n^{1/\beta}(\log n)^{-3(\alpha-1)})^{\beta-1}}$$

$$\leqslant A_{\alpha,n}E|X|^\beta n^{-1/\alpha}(\log n)^{3(\alpha-1)(\beta-1)} \to 0. \tag{3.4}$$

Thirdly, by the definitions of a'_{ni} and $\bar{X}_i$, we can estimate $T_n^* - ET_n^*$ by

$$n^{-1/p}|T_n^* - ET_n^*| \leqslant \frac{2\sum_{i=1}^n |a_{ni}|^\alpha \{n^{1/\beta}/(\log n)^{3(\alpha-1)}\}}{n^{1/p}(n^{1/\alpha}\log^{-2} n)^{\alpha-1}}$$

$$= \frac{2}{n(\log n)^{\alpha-1}} \sum_{i=1}^n |a_{ni}|^\alpha \to 0. \tag{3.5}$$

Finally, let $Y_{ni} = \bar{a}_{ni}(X_i - X_i^c)$. By $1/p = 1/\alpha + 1/\beta$, $1 < p < 2$, we have

$$E\sum_{i=1}^n Y_{ni}^2 \leqslant nA_{\alpha\wedge 2,n}^{\alpha\wedge 2}(n^{1/\alpha}\log^{-2} n)^{(2-\alpha)^+} n^{(2-\beta)^+/\beta}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}$$

$$= \mathrm{O}(\max\{n^{2/\alpha},\ n^{2/\beta},\ n\}).$$

On the other hand, for any $t > 0$, $tn^{1/p}|Y_{ni}| \leqslant tn^{2/p}\log^{-2} n$ and $\max\{n^{2/\alpha},\ n^{2/\beta},\ n\} = \mathrm{o}(n^{2/p}\log^{-2} n)$. It follows by Bernstein's inequality that

$$P(|\bar{T}_n - E\bar{T}_n| > tn^{1/p}) = P\left(\left|\sum_{i=1}^n (Y_{ni} - EY_{ni})\right| > tn^{1/p}\right)$$

$$\leqslant 2\exp\left(\frac{-t^2n^{2/p}}{tn^{2/p}\log^{-2} n}\right), \tag{3.6}$$

which is summable in n. The proof of the sufficiency part of Theorem 2.1 is complete in view of (3.1) thru (3.6).

3.2. Proof of the necessity part of Theorem 2.1

Suppose (2.2) is true for any weights sequence satisfying (2.1). Choose, for each $n, a_{n1} = \cdots, a_{n,n-1} = 0$ and $a_{nn} = n^{1/\alpha}$. Then, by (2.2) we have

$$n^{-1/\beta}X_n \to 0 \quad \text{a.s.},$$

which implies that $E(|X|^{\beta}) < \infty$. Since $\beta > 1$, EX exists. Moreover, by the sufficiency part of Theorem 2.1, we have

$$n^{-1/p}(T_n - ET_n) \to 0 \quad \text{a.s.}$$

for any weight sequence satisfying (2.1). Therefore,

$$n^{-1/p}ET_n = n^{-1/p}\sum_{i=1}^{n} a_{ni}EX \to 0$$

for any weight sequence $\{a_{ni}\}$ satisfying (2.1). Selecting $a_{ni} = 1$, we show that $EX = 0$. The necessity part is also proved, concluding Theorem 2.1.

3.3. *Proof of Theorem 2.2., Case 1,* $0 < \alpha \leqslant 1$.

We first prove the case $0 < \alpha \leqslant 1$. For convenience, we relabel (2.4) to be

$$\limsup |T_n|/b_n \leqslant h^{-1/\gamma}A_\alpha \quad \text{a.s.} \tag{3.7}$$

In this case, $b_n = n^{1/\alpha}\log^{1/\gamma} n$ and $\max_{1\leqslant i\leqslant n}|a_{ni}| \leqslant n^{1/\alpha}A_{\alpha,n}$, hence $\sum |a_{ni}| \leqslant n^{1/\alpha}A_{\alpha,n}$. Thus, we have

$$\limsup |T_n|/b_n \leqslant \limsup A_{\alpha,n}\log^{-1/\gamma} n \max_{i\leqslant n}|x_i| \leqslant A_\alpha h^{-1/\gamma}$$

which proves (3.7). Here, the last inequality follows from condition (2.3) that

$$\limsup_{n\to\infty}(\log n)^{-1/\gamma}\max\{|X_1|,\ldots,|X_n|\} \leqslant h^{-1/\gamma} \quad \text{a.s.} \tag{3.8}$$

The proof of (3.8) directly follows by calculating the probability of the events $\{(\log n)^{-1/\gamma}\ |X_n| > th^{-1/\gamma}\}$, $t > 1$, using the Markov inequality, together with the Borel–Cantelli Lemma. Note that (3.8) is true independent of α, $0 < \alpha < 2$.

For the converse, suppose that (3.7) holds for any choice of weights satisfying (2.1). Take $a_{ni} = 0$ for $i < n$ and $a_{nn} = n^{1/\alpha}A_\alpha$, then we have

$$\limsup_n (h/\log n)^{1/\gamma}|X_n| \leqslant 1 \quad \text{a.s.}$$

This means, for any $\varepsilon > 0$,

$$P(\mathrm{e}^{h|X_n|^{\gamma}/(1+\varepsilon)} \geqslant n \quad \text{i.o.}) = 0.$$

By Borel–Cantelli lemma, we conclude that $E\exp(h'|X|^{\gamma}) < \infty$, where $h' = h/(1+\varepsilon)$. Thus, the necessity part of case 1 is proved. □

3.4. *Proof of Theorem 2.2., Case 2,* $1 < \alpha < 2$.

In what follows, we will prove that (2.5) holds for $b_n = n^{1/\alpha}(\log n)^{1+1/\gamma-1/\lambda}$, where λ is defined by $1 < \lambda = \alpha(1+\gamma)/(\alpha+\gamma) < \alpha < 2$, giving the same b_n of (2.5).

By (3.8), we find that with probability one there are at most finitely many n such that $|x_n| > th^{-1/\gamma}\log^{1/\gamma} n$. From this, we adapt the following truncations to the X-variables. For any fixed $t > 1$,

$$X_i^c = X_iI[|X_i| \geqslant t(h^{-1}\log n)^{1/\gamma}],$$
$$X_i' = X_iI[(\log n)^{\delta_1} \leqslant |X_i| < t(h^{-1}\log n)^{1/\gamma}],$$
$$X_i'' = X_iI[|X_i| < (\log n)^{\delta_1}] = X_i - X_i^c - X_i',$$

where $\delta_1 = (\alpha+\gamma)/[\alpha\gamma(\gamma+1)] = 1/\alpha + 1/\gamma - 1/\lambda = 1/\lambda\gamma < 1/\gamma$.

Correspondingly, the truncations of the weights are set in view of condition $A_\alpha < \infty$, such that

$$a'_{ni} = a_{ni} I[|a_{ni}| \geqslant n^{1/\alpha} \log^{-1/\alpha} n],$$
$$a''_{ni} = a_{ni} I[n^{1/\alpha} \log^{-\delta_2} n \leqslant |a_{ni}| < n^{1/\alpha} \log^{-1/\alpha} n],$$
$$a'''_{ni} = a_{ni} I[|a_{ni}| < n^{1/\alpha} \log^{-\delta_2} n],$$

where $\delta_2 > 1/\lambda$.

To continue with the proof, write

$$T_n/b_n = b_n^{-1} \left\{ \sum a_{ni} X_i^c + \sum a'_{ni} X''_i + \sum (a'_{ni} + a''_{ni}) X'_i + \sum a''_{ni} X''_i + \sum a'''_{ni} X_i^* \right\},$$

where $X_i^* = X_i - X_i^c$.

Accordingly, we claim that it suffices to establish the following Lemmas.

Lemma 3.1. $\left| b_n^{-1} \sum a_{ni} X_i^c \right| \to 0$ *a.s.*

Proof. Condition $A_\alpha < \infty$ specifies that $\max_{1 \leqslant i \leqslant n} |a_{ni}| \leqslant n^{1/\alpha} A_{\alpha,n}$ a.s. It follows that the l.h.s. is bounded by $(\log n)^{-1/\gamma} \sum_{i=1}^{N(\omega)} |X_i|$, where $N(\omega)$, being the last i such that $|X_i| > t(h^{-1} \log i)^{1/\gamma}$, is finite as commented after (3.8). Thus, Lemma 3.1 holds. □

Lemma 3.2. $\limsup_{n\to\infty} b_n^{-1} |\sum a'_{ni} X''_i| \leqslant A_\alpha^\alpha$ *a.s.*

Proof. By the definition of a'_{ni} and X''_i, we have

$$b_n^{-1} \left| \sum a'_{ni} X''_i \right| \leqslant b_n^{-1} \sum |a_{ni}|^\alpha n^{1/\alpha - 1} \log^{\delta_1 + 1 - 1/\alpha} n \leqslant A_{\alpha,n}^\alpha$$

since $\delta_1 = 1/\alpha + 1/\gamma - 1/\lambda$. The lemma is proved. □

Lemma 3.3. $b_n^{-1} \sum (a'_{ni} + a''_{ni}) X'_i \to 0$ *a.s.*

Proof. We have, by Hölder's inequality and the bound of X_i noted in the proof of Lemma 3.1,

$$b_n^{-1} \left| \sum (a'_{ni} + a''_{ni}) X'_i \right| \leqslant t h^{-1/\gamma} A_{\alpha,n} M_n^{1-1/\alpha} \log^{-1+1/\lambda} n,$$

where $M_n = \sharp\{i \leqslant n, (a'_{ni} + a''_{ni}) X'_i \neq 0\}$.

Therefore, to complete the proof of Lemma 3.3, it suffices to show that for any $\varepsilon > 0$,

$$P(M_n^{1-1/\alpha} \geqslant \varepsilon (\log n)^{1-1/\lambda}, \text{ i.o.}) = 0. \tag{3.9}$$

By condition (2.3) $\sharp\{i \leqslant n,\ a'_{ni} + a''_{ni} \neq 0\} \leqslant A_{\alpha,n}^\alpha \log^{\alpha\delta_2} n$, and for some $C > 0$,

$$\begin{aligned} P\{M_n^{1-1/\alpha} \geqslant \varepsilon (\log n)^{1-1/\lambda}\} &= P\{M_n \geqslant \varepsilon' (\log n)^{\gamma/(\gamma+1)}\} \\ &\leqslant C (\log^{\alpha\delta_2} n P\{|X_1| > (\log n)^{\delta_1}\})^{\varepsilon (\log n)^{\gamma/(\gamma+1)}} \\ &\leqslant C \exp(-\varepsilon (\log n)^{\gamma/(\gamma+1)} \{h (\log n)^{\delta_1 \gamma} - \alpha\delta_2 \log\log n\}) \\ &\leqslant C \exp(-\tfrac{1}{2} \varepsilon h (\log n)^{1+\gamma/\alpha(\gamma+1)}) \end{aligned}$$

which is summable. Therefore, (3.9) holds and so does Lemma 3.3. □

Lemma 3.4. $b_n^{-1}|\sum a''_{ni}X''_i| \leqslant 3$ *a.s. and* $b_n^{-1}\sum a'''_{ni}X^*_i \to 0$ *a.s.*

Proof. We first show that $b_n^{-1}\sum a''_{ni}EX'_i \to 0$ and $b_n^{-1}\sum a'''_{ni}EX^*_i \to 0$.

Since $EX = 0$, by Markov inequality and condition (2.3) we have for n large

$$|EX''_i| = |EXI[|X| > \log^{\delta_1} n]| \leqslant (\log^{\delta_1} n)\exp(-h(\log n)^{\gamma\delta_1}).$$

From this and by noting that $\sharp\{i \leqslant n; a''_{ni} \neq 0\} = \mathrm{O}(\log^{\delta_2\alpha} n)$, we have

$$\left|b_n^{-1}\sum a''_{ni}EX''_i\right| \leqslant \mathrm{O}(\log^{\delta_1+\delta_2\alpha-1+1/\lambda-1/\gamma} n)\exp(-h(\log n)^{\gamma\delta_1}) \to 0.$$

Similarly, we have

$$|EX^*_i| = |EXI[|X| > th^{-1/\gamma}\log^{1/\gamma} n]| \leqslant th^{-1/\gamma}\log^{1/\gamma} n\exp(-t^\gamma \log n)$$

which implies that $b_n^{-1}\sum a'''_{ni}EX^*_i \to 0$, due to $t > 1$.

To complete the proof of this lemma, we will apply the Bernstein inequality to the random components. We first note that

$$b_n^{-1}|a''_{ni}X''_i| \leqslant (\log n)^{-1+1/\lambda-1/\gamma-1/\alpha+\delta_1} = 1/\log n, \tag{3.10}$$

and,

$$\begin{aligned}\mathrm{Var}\left(b_n^{-1}\sum a''_{ni}X''_i\right) &\leqslant b_n^{-2}\sum |a''_{ni}|^2E(X^2)\\ &\leqslant A^\alpha_{\alpha,n}\sigma^2(\log n)^{-2-2/\gamma+2/\lambda-(2-\alpha)/\alpha}\\ &\leqslant A^\alpha_{\alpha,n}\sigma^2(\log n)^{-1-2\delta_1} = \mathrm{o}(1/\log n),\end{aligned}$$

where $\sigma^2 = EX_1^2$. By (3.10), an application of the Bernstein inequality yields, for any $t > 3$,

$$P\left\{b_n^{-1}\left|\sum a''_{ni}(X''_i - EX''_i)\right| \geqslant t\right\} \leqslant 2\mathrm{e}^{-t\log n}.$$

The r.h.s of the above inequality is summable, which implies that

$$\limsup_{n\to\infty} b_n^{-1}\left|\sum a''_{ni}(X''_i - EX''_i)\right| \leqslant 3 \quad \text{a.s.}$$

The first assertion of Lemma 3.4 is proved.

Next, one finds that for some constants $c > 1$,

$$b_n^{-1}|a'''_{ni}X^*_i| \leqslant th^{-1/\gamma}\log^{-1+1/\lambda-\delta_2} n = \mathrm{o}(\log^{-c} n) \tag{3.11}$$

since $\delta_2 > 1/\lambda$. Moreover,

$$\begin{aligned}\mathrm{Var}\left(b_n^{-1}\sum a'''_{ni}X^*_i\right) &\leqslant b_n^{-2}\sum |a'''_{ni}|^2\sigma^2\\ &\leqslant b_n^{-2}\sum |a_{ni}|^\alpha(n^{1/\alpha}\log^{-\delta_2} n)^{2-\alpha}\sigma^2\\ &\leqslant A^\alpha_{\alpha,n}(\log n)^{-2-2/\gamma+2/\lambda-(2-\alpha)\delta_2}\sigma^2\\ &= \mathrm{o}(\log^{-c} n),\end{aligned} \tag{3.12}$$

where the last estimate follows from that $\delta_2 > 1/\alpha$, $\delta_1 = 1/\gamma + 1/\alpha - 1/\lambda$, and

$$-1 - 2/\gamma + 2/\lambda - (2-\alpha)\delta_2 = (2-\alpha)\left(\frac{1}{\alpha} - \delta_2\right) - 2\delta_1 < 0.$$

By (3.11) and (3.12), an application of Bernstein's inequality yields that for any $\varepsilon > 0$

$$P\left\{b_n^{-1}\left|\sum a_{ni}'''(X_i^* - EX_i^*)\right| \geqslant \varepsilon\right\} \leqslant 2\mathrm{e}^{-\varepsilon^2 \log^c n}.$$

The r.h.s above is likewise summable, implying $b_n^{-1}\sum a_{ni}'''(X_i^* - EX_i^*) \to 0$ a.s. The second assertion of Lemma 3.4 is also proved, concluding Lemma 3.4.

Combine Lemmas 3.1–3.4, it follows immediately that

$$\limsup_n b_n^{-1}\left|\sum a_{ni}X_i\right| \leqslant A_\alpha^\alpha + 3 \quad \text{a.s.}$$

To claim the sufficiency part, (2.5) of Theorem 2.2, we will improve the last estimate making the right-hand side above be zero. Define $\tilde{a}_{ni} = \eta a_{ni}$ and $\tilde{X}_i = wX_i$. Then, by what we have proved,

$$\limsup_n b_n^{-1}\left|\sum \tilde{a}_{ni}\tilde{X}_i\right| \leqslant \tilde{A}_\alpha^\alpha + 3 = w\eta^\alpha A_\alpha^\alpha + 3$$

which is equivalent to

$$\limsup_n b_n^{-1}\left|\sum a_{ni}X_i\right| = \eta^{\alpha-1}A_\alpha^\alpha + 3/w.$$

Let $\eta \to 0$ and $w \to \infty$, our assertions hold, thus concluding the proof of (2.5). The necessity part of case 2 simply follows by the same argument as that of case 1. Therefore, the proof of Theorem 2.2 is complete. □

References

Bai, Z.D., Cheng, P.E., Zhang, C.H., 1997. An extension of the Hardy–Littlewood strong law. Statist. Sinica, 923–928.
Cheng, P.E., 1995. A note on strong convergence rates in nonparametric regression. Statist. Probab. Lett. 24, 357–364.
Cuzick, J., 1995. A strong law for weighted sums of i.i.d. random variables. J. Theoret. Probab 8, 625–641.